Prof. Dr. Mark. Podolskij

Blatt 4

Mo 14-16 Uhr

Tutor: Samuel Kiban

Jannis Schnitzer, Themistokles Parassis

1	2	3	4	Σ
0,5	4	3,5	4	12

Hausaufgaben

1. Sei $f \in C([a,b])$, α monoton wachsend auf $[a,b]$ und

$$F(x) := \int_a^x f(t)d\alpha(t) \qquad \text{für } x \in [a,b]$$

Zeigen Sie:

a) Ist α stetig in $x_0 \in [a,b]$, so ist auch F dort stetig.

b) Falls α differenzierbar ist, dann ist auch F differenzierbar mit $F'(x) = f(x)\alpha'(x)$.

2. Es seien $f, g : \mathbb{R} \to \mathbb{R}$ Riemann-integrierbare, 2π-periodische Funktionen. Wir bezeichnen die Fourier-Koeffizienten von f mit a_n, b_n und diejenigen von g mit α_n, β_n. Zeigen Sie eine etwas allgemeinere Version der Parseval'schen Gleichung (Satz 16.7 (2)), nämlich

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx = \frac{1}{2}a_0\alpha_0 + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k).$$

3. Bestimmen Sie die Fourier-Reihen der folgenden Funktionen

a) $f : \mathbb{R} \to \mathbb{R}, x \mapsto |\sin x|$.

b) $f : \mathbb{R} \to \mathbb{R}, x \mapsto (\cos x)^k$ für $k \in \mathbb{N}$.

4. Es sei $f : \mathbb{R} \to \mathbb{R}$ eine stetig differenzierbare, 2π-periodische Funktion. Weiterhin sei $\alpha \in \mathbb{R}$, so dass α/π irrational ist. Zeigen Sie, dass für alle $x \in \mathbb{R}$ gilt

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt.$$

Hinweis: Zeigen Sie die Behauptung zuerst für die Funktionen $\cos(kx)$, $\sin(kx)$.

Abgabe der Hausaufgaben: Mittwoch, den 16.05. um 9:15 Uhr in den Briefkästen im Foyer des INF 294.

Nr 1

0,5 a) Wenn F stetig in Punkt $x_0 \Leftrightarrow \lim\limits_{x \to x_0} F(x) = F(x_0)$

$$|F(x) - F(x_0)| = \left| \int_a^x f(t)\, d\alpha(t) - \int_a^{x_0} f(t)\, d\alpha(t) \right|$$

$$= \left| \left(\int_a^{x_0} f(t)\, d\alpha(t) + \int_{x_0}^{x} f(t)\, d\alpha t \right) - \int_a^{x_0} f(t)\, d\alpha(t) \right|$$

$$= \left| \int_{x_0}^{x} f(t)\, d\alpha(t) \right|$$

Sei a stetig im Punkt $x_0 \in [a, b] \Rightarrow \lim\limits_{x \to x_0} a(x) = a(x_0)$

$$\cancel{\approx} \left| \sum_{i=0}^{n} f(\xi_i) \cdot (a(x_i) - a(x_{i-1})) \right| \leq \sum_{i=1}^{n} |f(\xi_i) \cdot (a(x_i) - a(x_{i-1}))|$$

$$= \sum_{i=1}^{n} \underbrace{|f(\xi_i)|}_{\leq \max\limits_{x_0 \leq \xi \leq x} |f(\xi)| =: A} \cdot |a(x_i) - a(x_{i-1})|$$

Was sind die x_i und ξ_i ?

$\cdot (a(x_0) - a(x))$?

$$\cancel{\leq} A \cdot \sum_{i=1}^{n} \underbrace{|a(x_i) - a(x_{i-1})|}_{\geq 0, \text{ da } x_{i-1} < x_i}$$

?

a.min

$$\Rightarrow a(x_{i-1}) \leq a(x_i) \Rightarrow a(x_i) - a(x_{i-1}) \geq 0$$

$$\cancel{\leq} A \cdot \sum_{i=1}^{n} [a(x_i) - a(x_{i-1})]$$

$$= A\left[(a(x_1) - a(x_0)) + (a(x_2) - a(x_1)) + \ldots + (a(x_n) - a(x_{n-1}))\right]$$

$$= A[a(x_n) - a(x_0)] = A[\underbrace{a(x) - a(x_0)}_{\to 0 \text{ für } x \to x_0, \text{ da } a \text{ stetig im Pkt. } x_0}], \text{ da } x_n := x$$

Aha

?

$$0 \leq |F(x) - F(x_0)| \leq A \cdot [\underbrace{a(x) - a(x_0)}_{\to 0 \text{ für } x \to x_0}]$$

$\Rightarrow |F(x) - F(x_0)| \to 0$ für $x - x_0 \Rightarrow F(x) \to F(x_0)$ für $x \mapsto x_0$

$\Rightarrow F$ stetig im Punkt. x_0. ∎

Vorsicht!

b) Sei $\alpha(t)$ diff.bar $\forall\ x \in [a,b]$

$F'(x) = f(x) \cdot \alpha'(x)$

aber α' nicht stetig!

Wenn $\alpha(t)$ diff.bar ist, so ist $\alpha(t)$ stetig:

$\Rightarrow F(x) := \int_a^x f(t)\, d\alpha(t) \not= \int_a^x f(t) \cdot \alpha'(t)$ (dt) Nein.

Differenzenquotienten aufstellen:

$$\frac{\int_a^x (f(t) \cdot \alpha'(t)) - \int_a^{x_0} (f(t)\alpha'(t))}{x - x_0} = \frac{\Delta\left(\int_a^x f(t)\, \alpha'(t)\right)}{\Delta x} \quad ?$$

$$\xRightarrow{\Delta x \to 0} \cancel{\int_a^x} f(x) \cdot \alpha'(x) \cdot \frac{1}{\cancel{\Delta x}} = f(x) \cdot \alpha'(x) = F'(x) \quad \blacksquare$$

???

Nee

0,5/4

$$2.\ \frac{1}{4}\left[(f+g)^2-(f-g)^2\right] = f\cdot g$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad \alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad \alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad \beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)\,dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{4}\left[(f+g)^2-(f-g)^2\right]dx$$

(Schreibe: F. f, g statt f(x), g(x))

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{4}(f+g)^2dx - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{4}(f-g)^2dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}f+\frac{1}{2}g\right)^2dx - \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}f-\frac{1}{2}g\right)^2dx$$

$$\overset{?}{\underset{16.7(2)}{=}} \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(f+g)\,dx\right)^2 + \sum_{k=1}^{\infty}\left(\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(f+g)\cdot\cos(nx)\,dx\right)^2+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(f+g)\sin(nx)\,dx\right)^2\right)$$

$$-\frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(f-g)\,dx\right)^2 - \sum_{k=1}^{\infty}\left(\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(f-g)\cos(nx)\,dx\right)^2+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(f-g)\sin(nx)\,dx\right)^2\right)$$

Linearität des Integrals; dann 1. und 2. Binomische Formel anwenden.

$$\cancel{\frac{1}{2\pi^2}\left(\int_{-\pi}^{\pi}\frac{1}{2}f\,dx\right)^2+\frac{1}{2\pi^2}\left(\int_{-\pi}^{\pi}\frac{1}{2}g\,dx\right)^2+}$$

$$= \frac{1}{8\pi^2}\left(\int_{-\pi}^{\pi}f\,dx\right)^2+\frac{1}{8\pi^2}\left(\int_{-\pi}^{\pi}g\,dx\right)^2+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}f\,dx\int_{-\pi}^{\pi}g\,dx$$

$$+\sum_{k=1}^{\infty}\Bigg[\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}f\cos(nx)\,dx\right)^2+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}g\cdot\cos(nx)\,dx\right)^2+\frac{1}{2\pi^2}\int_{-\pi}^{\pi}f\cos(nx)\,dx\int_{-\pi}^{\pi}g\cdot\cos(nx)\,dx$$

$$+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}f\sin(nx)\,dx\right)^2+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}g\cdot\sin(nx)\,dx\right)^2+\frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f\sin(nx)\,dx\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g\sin(nx)\,dx\right)\Bigg]$$

$$-\frac{1}{8}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f\,dx\right)^2-\frac{1}{8}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g\,dx\right)^2+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}f\,dx\cdot\int_{-\pi}^{\pi}g\,dx$$

$$-\sum_{k=1}^{\infty}\Bigg[\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}f\cos(nx)\,dx\right)^2+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}g\cos(nx)\,dx\right)^2-\frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f\cos(nx)\,dx\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g\cdot\cos(nx)\,dx\right)$$

$$+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}f\cdot\sin(nx)\,dx\right)^2+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}g\cdot\sin(nx)\,dx\right)^2-\frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f\sin(nx)\,dx\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g\sin(nx)\,dx\right)\Bigg]$$

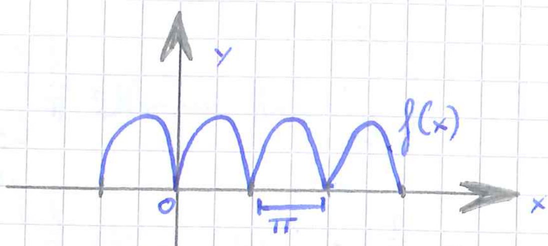
$$=\frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx\right)\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\,dx\right)+\sum_{k=1}^{\infty}\Bigg[\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\cos(nx)\,dx\right)+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\sin(nx)\,dx\right)\Bigg]$$

$$= \frac{1}{2} a_0 \cdot \alpha_0 + \sum_{n=1}^{\infty} \left(a_n \cdot \alpha_n + b_n \cdot \beta_n \right) \quad \square$$

4/4

Nr 3

1,5 a) $f(x) = |\sin x| \quad , n \in \mathbb{N}_0$



i) $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \, dx$

gerade Fkt. ⟶ zum Ursprung symm.

$\Rightarrow a_0 = \frac{2}{\pi} \int_0^{\pi} \underbrace{|\sin(x)|}_{\geq 0} dx$, da $0 \leq x \leq \pi$

$\Rightarrow a_0 = \frac{2}{\pi} \int_0^{\pi} \sin(x) \, dx = \frac{2}{\pi} \left[-\cos(x) dx\right]_0^{\pi} = \frac{2}{\pi}\left(-\cos(\pi) - (-\cos(0))\right) + \cancel{c}$

$= \frac{2}{\pi}\left(1 - (-1)\right)_{+} = \frac{2}{\pi} \cdot 2 = \underline{\underline{\frac{4}{\pi}}} + \cancel{c}$

ii) $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(kx) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \overbrace{|\sin(x)|}^{\text{gerade}} \overbrace{\cos(nx)}^{\text{gerade}} dx = \frac{2}{\pi} \int_0^{\pi} \underbrace{|\sin(x)|}_{\geq 0} \cos(nx) \, dx$

$= \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(nx) \, dx \mid I := \sin(x) \cdot \cos(nx) \, dx$

$f'(x) = \sin(x) \qquad g(x) = \cos(nx)$

$f(x) = -\cos(x) \qquad g'(x) = -n \sin(nx)$

$\int f'(x) g(x) = \left[f(x) \cdot g(x)\right] - \int_0 f(x) \cdot g'(x) \, dx$

$\Rightarrow \left[-\cos(x) \cdot \cos(nx)\right]_0^{\pi} - n \int_0^{\pi} \cos(x) \sin(nx) \, dx$

$\cancel{\frac{2}{\pi}} \int_0^{\pi} I = \left[-\cos(x) \cdot \cos(nx)\right]_0^{\pi} - n \int_0^{\pi} f(x) \cdot g'(x) \, dx$

$f'(x) = \cos(x) \qquad g(x) = \sin(nx)$

$f(x) = +\sin(x) \qquad g'(x) = +n\cos(nx)$

$\Rightarrow \cancel{\frac{2}{\pi}} \int_0^{\pi} I = \left[-\cos(x) \cdot \cos(nx)\right]_0^{\pi} - n\left(\left[\pm\sin(x) \cdot \sin(nx)\right]_0^{\pi} - n \int \overbrace{\sin(x) \cos(nx) \, dx}^{I}\right)$

$\Leftrightarrow \left(\cancel{\frac{2}{\pi}}^{\,1} \cancel{+}^{\,-} n^2\right) \int_0^{\pi} I = \left[-\cos(x)\cos(nx)\right]_0^{\pi} + n \overbrace{\left[\sin(x) \sin(nx)\right]_0^{\pi}}^{=0}$

$\Leftrightarrow \int_0^{\pi} I = \frac{\left[-\cos(x) \cos(nx)\right]_0^{\pi}}{\left(\cancel{\frac{2}{\pi} + n^2}\right)_{1-n^2}} = \frac{\left[-\cos(\pi) \cdot \cos(n \cdot \pi) - \left(-\cos(0) \cdot \cos(n \cdot 0)\right)\right]}{\left(\cancel{\frac{2}{\pi} + n^2}\right)_{1-n^2}}$

$= \frac{+1 \cdot (-1)^n - (-1)}{\left(\cancel{\frac{2}{\pi} + n^2}\right)_{1-n^2}} = \frac{(-1)^n + 1}{\left(\cancel{\frac{2}{\pi} + n^2}\right)_{1-n^2}} \begin{cases} n \text{ gerade} = \frac{2}{\cancel{\left(\frac{2}{\pi} + n^2\right)}} \\ n \text{ ungerade} = 0 \end{cases} \quad \frac{-4}{(n^2-1)\pi}$

iii) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)| \circ \sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} |\sin(x)|\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\sin(nx)\,dx$

$= \frac{-1}{\pi}\int_{-\pi}^{0} + \sin(x)\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin(x)\sin(nx)\,dx$

1.) $\frac{1}{\pi}\int \sin(x)\sin(nx)\,dx$

$f'(x) = \sin(x) \qquad g(x) = \sin(nx)$

$f(x) = -\cos(x) \qquad g'(x) = n\cos(nx)$

$\Rightarrow \frac{1}{\pi}\int \overbrace{\sin(x)\sin(nx)}^{I}\,dx = \left[-\cos(x)\sin(nx)\right]_a^b + n\int \cos(x)\cos(nx)\,dx$

$f'(x) = \cos(x) \qquad g(x) = \cos(nx)$

$f(x) = -\sin(x) \qquad g'(x) = n\sin(nx)$

$\Rightarrow \frac{1}{\pi}\int I = \left[-\cos(x)\sin(nx)\right]_a^b + n\left(\left[-\sin(x)\cdot\cos(nx)\right]_a^b - n\int \sin(x)\sin(nx)\,dx\right)$

$\Leftrightarrow \left(\frac{1}{\pi}+n^2\right)\int I = \left[-\cos(x)\sin(nx)\right]_a^b + n\left[-\sin(x)\cos(nx)\right]_a^b$

$\Leftrightarrow \int I = \frac{\left[-\cos(x)\sin(nx)\right]_a^b + n\left[-\sin(x)\cos(nx)\right]_a^b}{\left(n^2+\frac{1}{\pi}\right)}$

$\Rightarrow b_n = \frac{\left[-\cos(x)\sin(nx)\right]_{-\pi}^{0} + n\left[-\sin(x)\cos(nx)\right]_{-\pi}^{0}}{\left(n^2-\frac{1}{\pi}\right)} + \frac{\left[-\cos(x)\sin(nx)\right]_0^{\pi} + n\left[-\sin\cos(nx)\right]_0^{\pi}}{\left(n^2+\frac{1}{\pi}\right)}$

$= \frac{\left[(-\cos(0)\sin(n\cdot 0) - (-\cos(-\pi)\cdot\sin(n\cdot(-\pi))\right] + n\left[-\sin(0)\cos(n\cdot 0) - (-\sin(-\pi)\cdot\cos(n(-\pi))\right]}{\left(n^2-\frac{1}{\pi}\right)}$

$+ \frac{\left[-\cos(\pi)\sin(n\cdot\pi) - (-\cos(0)\sin(n\cdot 0)\right] + n\left[-\sin(\pi)\cos(\pi\cdot n) - (-\sin(0)\cos(\pi\cdot 0))\right]}{\left(n^2+\frac{1}{\pi}\right)}$

$= 0$ ✓ (folgt aus ungerade)

$\Rightarrow$ Fourierreihe:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

$$\Rightarrow \frac{2}{\pi} + \sum_{n=1}^{\infty}\left(\left(\frac{(-1)^n+1}{(\frac{2}{\pi}+n^2)}\right)\cdot\cos(nx)\right)$$

1.) n gerade $\Rightarrow$ $\alpha := \frac{2}{\pi} + \sum_{n=2i}^{\infty}\left(\frac{2}{(\frac{2}{\pi}+n^2)}\right)\cdot\cos(nx)$

2.) n ungerade $\Rightarrow$ $K = \frac{2}{\pi}$

Nr 3

1 b) $f(x) = (\cos(x))^k$, $k \in \mathbb{N}$, $n \in \mathbb{N}_0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\cos(x))^k \, dx \quad | \, (\cos(x))^k = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\cos((k-2i)x)$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\cos((k-2i)x)\right) dx$$

$$= \frac{1}{2^k\pi}\sum_{i=0}^{k}\binom{k}{i}\int_{-\pi}^{\pi}\underbrace{\cos((k-2i)x)}_{\text{gerade Fkt.}} \, dx$$

$$= \frac{1}{2^{k-1}\pi}\sum_{i=1}^{k}\binom{k}{i}\int_{0}^{\pi}\cos((k-2i)x) \, dx$$

$$= \frac{1}{2^{k-1}\pi}\sum_{i=1}^{k}\binom{k}{i}\left[\frac{\sin((k-2i)x)}{k-2i}\right]_0^{\pi}$$

$i = \frac{k}{2}$?

$$= \frac{1}{2^{k-1}\pi}\sum_{i=1}^{k}\binom{k}{i}\cdot\frac{\sin((k-2i)\cdot\pi)}{k-2i}, \quad k \neq 2i$$

$\Rightarrow$ Für beliebige $k, i \in \mathbb{N}$ ergeben sich ganze Perioden π für sin $\Rightarrow$ sin ist immer null! Und für $k = 2i$? Nicht 0!

$\Rightarrow \underline{\underline{a_0 = 0}}$

$$\text{i) } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\cos((k-2i)x)\right)\cdot\cos(nx) \, dx$$

$$\Leftrightarrow a_n = \frac{1}{2^k\pi}\cdot\sum_{i=0}^{k}\binom{k}{i}\int_{-\pi}^{\pi}{}^{*}\underbrace{\cos((k-2i)x)\cdot\cos(nx)}_{\text{gerade}} \, dx \quad | \, I := \cos((k-2i)x)\cdot\cos(nx)\,dx$$

$f'(x) = \cos(nx)$ $\qquad g(x) = \cos((k-2i)x)$

$f(x) = \frac{1}{n}\sin(nx)$ $\qquad g'(x) = -(k-2i)\sin((k-2i)x)$

$\Rightarrow$ ~~$\frac{1}{2^k\pi}\sum_i$~~

$* \quad a_n = \frac{1}{2^{k-1}\pi}\sum_{i=1}^{k}\int_0^{\pi} I$

3,5/4

$$\int f'(x) g(x) = [f(x) g(x)] - \int f(x) \cdot g'(x)$$

$$\Rightarrow \frac{1}{2^{k-1}\pi} \sum_{i=1}^{k} \binom{k}{i} \int_0^{\pi} I = \left[\frac{1}{n} \sin(nx) \cdot \cos((k-2i)x)\right]_0^{\pi} + \frac{1}{n} \int \sin(nx) \cdot (k-2i) \sin((k-2i)x)$$

$f'(x) = \sin(nx)$, $f(x) = -\frac{1}{n}\cos(nx)$; $g(x) = \sin((k-2i)x)$, $g'(x) = (k-2i) \cdot \cos((k-2i)x)$

$$= \left[\frac{1}{n} \sin(x) \cos((k-2i)x)\right]_0^{\pi} + \frac{(k-2i)}{n} \left[\left[-\frac{1}{n} \cos(nx) \sin((k-2i)x)\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{n} \cos(nx) \cdot (k-2i) \cdot \cos((k-2i)x)\right]$$

$$= \left[\frac{1}{n} \sin(x) \cos((k-2i)x)\right]_0^{\pi} - \frac{(k-2i)}{n^2} \left[\cos(nx) \sin((k-2i)x)\right]_0^{\pi} + \frac{(k-2i)^2}{n^2} \int \underbrace{\cos(nx) \cos((k-2i)x)}_{= I} dx$$

$$\Leftrightarrow \left(\frac{1}{2^{k-1}\pi} \sum_{i=1}^{k} \binom{k}{i} - \frac{(k-2i)^2}{n^2}\right) \int_0^{\pi} I = -\left[\cos(nx) \sin((k-2i)x)\right]_0^{\pi} + \left[\frac{1}{n} \sin(nx) \cos((k-2i)x)\right]_0^{\pi}$$

$$\Leftrightarrow \int_0^{\pi} I = \frac{-\left[\cos(nx) \sin((k-2i)x)\right]_0^{\pi} + \left[\frac{1}{n} \sin(nx) \cos((k-2i)x)\right]_0^{\pi}}{\left(\frac{1}{2^{k-1}\pi} \sum_{i=1}^{k} \binom{k}{i} - \frac{(k-2i)^2}{n^2}\right)}$$

$$= \frac{-\left(\overset{1}{\cos(n\pi)} \overset{0}{\sin((k-2i)\pi)} - \left(\overset{1}{\cos(0)} \cdot \overset{0}{\sin(0)}\right)\right)}{\left(\frac{1}{2^{k-1}\pi} \sum_{i=1}^{k} \binom{k}{i} - \frac{(k-2i)^2}{n^2}\right)} + \frac{\left(\left(\frac{1}{n} \overset{0}{\sin(n\pi)} \cdot \overset{1}{\cos((k-2i)\pi)} - \left(\frac{1}{n} \overset{0}{\sin(0)} \cdot \overset{1}{\cos(0)}\right)\right)\right)}{\left(\frac{1}{2^{k-1}\pi} \sum_{i=1}^{k} \binom{k}{i} - \frac{(k-2i)^2}{n^2}\right)}$$

$$= \underline{\underline{0}}$$ Nein!

ii) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx = \frac{1}{2^k \pi} \sum_{i=0}^{k} \binom{k}{i} \int_{-\pi}^{\pi} \{\cos((k-2i)x) \sin(nx) dx\} =: I$

$$\frac{1}{2^k \pi} \sum_{i=0}^{k} \binom{k}{i} \int_{-\pi}^{\pi} I = \left[f(x) g(x)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x) g'(x) dx$$

$f'(x) = \sin(nx)$ $\quad$ $g(x) = \cos((k-2i)x)$

$f(x) = -\frac{1}{n}\cos(nx)$ $\quad$ $g'(x) = -(k-2i) \sin((k-2i)x)$

$$\frac{1}{2^k\pi}\sum_{i=0}^{k}\binom{k}{i}\int_{-\pi}^{\pi} I = \left[-\frac{1}{n}\cos(nx)\cos((k-2i)x)\right]_{-\pi}^{\pi} - \frac{(k-2i)}{n}\int_{-\pi}^{\pi}\cos(nx)\sin((k-2i)x)\,dx$$

$$f'(x) = \cos(nx) \qquad g(x) = \sin((k-2i)x)$$

$$f(x) = \frac{1}{n}\sin(nx) \qquad g'(x) = (k-2i)\cos((k-2i)x)$$

$$= \left[-\frac{1}{n}\cos(nx)\cos((k-2i)x)\right]_{-\pi}^{\pi} - \frac{(k-2i)}{n}\left\{\left[\frac{1}{n}\sin(nx)\cdot\sin((k-2i)x)\right]_{-\pi}^{\pi} - \frac{(k-2i)}{n}\int_{-\pi}^{\pi}\sin(nx)\cos((k-2i)x)\,dx\right.$$

$$\Longleftrightarrow \left(\frac{1}{2^k\pi}\sum_{i=0}^{k}\binom{k}{i} - \frac{(k-2i)^2}{n^2}\right)\int_{-\pi}^{\pi} I = \left[-\frac{1}{n}\cos(nx)\cos((k-2i)x)\right]_{-\pi}^{\pi} - \frac{(k-2i)}{n^2}\left[\sin(nx)\sin((k-2i)x)\right]_{-\pi}^{\pi}$$

$$\Longleftrightarrow \int_{-\pi}^{\pi} I = \frac{\left[\overbrace{\cos(n\pi)\cdot\cos((k-2i)\pi) - (\cos(-n\pi)\cdot\cos((k-2i)-\pi)}^{1}\right]}{n\left(\frac{(k-2i)^2}{n^2} - \frac{1}{2^k\pi}\sum_{i=0}^{k}\binom{k}{i}\right)}$$

$$- \frac{(k-2i)\left[\sin(n\pi)\cdot\sin((k-2i)\pi) - (\sin(-n\pi)\cdot\sin((k-2i)-\pi))\right]}{n^2\left(\frac{(k-2i)^2}{n^2} - \frac{1}{2^k\pi}\sum_{i=0}^{k}\binom{k}{i}\right)}$$

$$= \frac{(-1)^n\cdot(-1)^{(k-2i)} + 1\cdot(-1)^{(k-2i)}}{n\left(\frac{(k-2i)}{n^2} - \frac{1}{2^k\pi}\sum_{i=0}^{k}\binom{k}{i}\right)} = \frac{((-1)^n+1)\cdot(-1)^{(k-2i)}}{n\left(\frac{(k-2i)}{n^2} - \frac{1}{2^k\pi}\sum_{i=0}^{k}\binom{k}{i}\right)}, \quad n \neq 0$$

$$= \begin{cases} \xrightarrow{n \text{ ungerade}} 0 \\ n \text{ gerade}, (k-2i) \text{ gerade} \Rightarrow \dfrac{2}{n\left(\frac{(k-2i)}{n^2} - \frac{1}{2^k\pi}\sum_{i=0}^{n}\binom{k}{i}\right)} \\ n \text{ gerade}, (k-2i) \text{ ungerade} \Rightarrow \dfrac{-2}{n\left(\frac{(k-2i)}{n^2} - \frac{1}{2^k\pi}\sum\binom{k}{i}\right)} \end{cases}$$

Nah!
Immer 0, da ungerade!

$$\Rightarrow \text{Fourier} \sum_{n=1}^{\infty}\left(\frac{2\cdot(-1)^{(2k-1)}}{n\left(\frac{k-2i}{n^2} - \frac{1}{2^k\pi}\sum\binom{k}{i}\right)}\right)\cdot\sin(nx)$$

4. Zunächst $\cos(kx)$, $\sin(kx)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(kt)\,dt = \frac{1}{2\pi}\left[\frac{1}{k}\sin(kt)\right]_{-\pi}^{\pi} = 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(kt)\,dt = \frac{1}{2\pi}\left[-\frac{1}{k}\cos(kt)\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(-\frac{1}{k}\cos(k\pi)+\frac{1}{k}\cos(-k\pi)\right)$$

$$\overset{\text{Symmetrie}}{=} \frac{1}{2\pi}\left(+\frac{1}{k}\cos(k\pi)-\frac{1}{k}\cos(k\pi)\right) = 0.$$

Betrachte nun:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(kx+kn\alpha) \overset{\text{Add.theoreme}}{=} \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{\infty}\left(\cos(kx)\cos(kn\alpha)-\sin(kx)\sin(kn\alpha)\right)$$

$$= \cos(kx)\cdot\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(kn\alpha) - \sin(kx)\cdot\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sin(kn\alpha)$$

$$\cancel{\overset{\text{ZÜ3b}}{=} \cos(kx)\cdot\lim_{N\to\infty}\frac{1}{N}\left(\sin\left((N+\tfrac{1}{2})k\alpha\right)\right)}$$

$$\overset{\text{ZÜ3b}}{=} \cos(kx)\cdot\lim_{N\to\infty}\frac{1}{N}\underbrace{\left[\frac{\sin\left((N+\frac{1}{2})k\alpha\right)}{2\sin\left(\frac{1}{2}k\alpha\right)}-\frac{1}{2}\right]}_{\substack{\text{Beschränkt durch ein}\\ r\in\mathbb{R}\text{, denn}\\ -1\leq\sin(x)\leq 1\ \forall x\in\mathbb{R}}} - \sin(kx)\cdot\lim_{N\to\infty}\frac{1}{N}\cdot\underbrace{\left[\frac{\cos\left(\frac{1}{2}k\alpha\right)-\cos\left((N+\frac{1}{2})k\alpha\right)}{2\sin\left(\frac{1}{2}k\alpha\right)}\right]}_{\substack{\text{Beschränkt durch ein}\\ s\in\mathbb{R}\text{, denn}\\ -1\leq\cos(x)\leq 1\text{ und}\\ -1\leq\sin(x)\leq 1\ \forall x\in\mathbb{R}.}}$$

$$\Rightarrow\ = \cos(kx)\cdot\lim_{N\to\infty}\frac{1}{N}\cdot f_N(x) - \sin(kx)\cdot\lim_{N\to\infty}\frac{1}{N}\,g_N(x)$$

mit $f_N(x)$, $g_N(x)$ beschränkt $\forall x, N\in\mathbb{R}$

$$= 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(kt)\,dt.$$

Betr. nun:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sin(kx+kn\alpha) \overset{\text{Add.theorem}}{=} \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\sin(kx)\cos(kn\alpha)+\cos(kx)\sin(kn\alpha)\right)$$

$$= \sin(kx)\underbrace{\lim_{N\to\infty}\sum_{n=1}^{N}\cos(kn\alpha)}_{=0} + \cos(kx)\underbrace{\lim_{N\to\infty}\sum_{n=1}^{N}\sin(kn\alpha)}_{=0\ \text{(s.o.)}}$$

$$= 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(kt)\,dt.$$

jetzt: $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left\{a_k\cos(kt)+b_k\sin(kt)\right\}dt$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}a_0\,dt + $ ~~[illegible]~~ $\sum_{k=1}^{\infty} a_k\int_{-\pi}^{\pi}\cos(kt)\,dt + b_k\int_{-\pi}^{\pi}\sin(kt)\,dt\}$

(Vertauschung von Summe und Integral erlaubt, denn:

$$\int_{\tilde{x}_0}^{\tilde{x}_1}\sum_{n=1}^{\infty}a_n(x)\,dx = \lim_{\nu\to\infty}\frac{\tilde{x}_1-\tilde{x}_0}{\nu}\cdot\sum_{k=1}^{\nu}\left\{(x_k-x_{k-1})\cdot\lim_{N\to\infty}\sum_{n=1}^{N}a_n(\xi_k)\right\}$$

~~[illegible]~~ und wenn $\sum_{n=1}^{\infty}a_n(x)$ glm. konv., so ist dies nach 11.5 (Anm 1) dasselbe wie

• $(x_k)_{k\in\mathbb{N}}$ Zerl. v. $[\tilde{x}_0,\tilde{x}_1]$
• ξ_k Zwp.

$$= \lim_{N\to\infty}\sum_{n=1}^{N}\lim_{\nu\to\infty}\frac{\tilde{x}_1-\tilde{x}_0}{\nu}\sum_{k=1}^{\nu}\left\{(x_k-x_{k-1})\cdot a_n(\xi_k)\right\}$$

$$= \sum_{n=1}^{\infty}\int_{\tilde{x}_0}^{\tilde{x}_1}a_n(x)\,dx$$)

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,dt = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left\{a_k\cdot\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(kx+kn\alpha) + b_k\lim_{N\to\infty}\sum_{n=1}^{N}\sin(kx+kn\alpha)\right\}$$

$$= \underbrace{\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{2}a_0}_{\text{ändert nichts, da }\frac{1}{2}a_0\text{ nicht von }n\text{ abhängig ist.}} + \sum_{k=1}^{\infty}\left\{\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left[a_k\cos\left(k(x+n\alpha)\right)+b_k\sin\left(k(x+n\alpha)\right)\right]\right\}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{2}a_0 + \lim_{K\to\infty}\sum_{k=1}^{K}\left\{\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left[a_k\cos\left(k(x+n\alpha)\right)+b_k\sin\left(k(x+n\alpha)\right)\right]\right\}$$

Nach 16.1 ist die Fourierreihe glm. konv. Daher gilt nach 11.5:

In 11.5 wurden 2 bestimmte GWe vertauscht. ~~[illegible]~~ Wie genau ist das hier anwendbar?

$$\overset{?}{=} \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{2}a_0 + \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lim_{K\to\infty}\sum_{k=1}^{K}\left\{a_k\cos\left(k(x+n\alpha)\right)+b_k\sin\left(k(x+n\alpha)\right)\right\}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\Bigg[\underbrace{\frac{1}{2}a_0+\sum_{k=1}^{\infty}\left\{a_k\cos\left(k(x+n\alpha)\right)+b_k\sin\left(k(x+n\alpha)\right)\right\}}_{\text{Fourierreihe}}\Bigg]$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(x+n\alpha) \quad \square$$

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