

Formelblatt

Vektor-Identitäten

$$\begin{aligned}\vec{A} \cdot (\vec{B} \times \vec{C}) &= \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \\ \vec{A} \times (\vec{B} \times \vec{C}) &= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \\ (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) &= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \\ \nabla \times \nabla \phi &= 0 \\ \nabla \cdot (\nabla \times \vec{A}) &= 0 \\ \nabla \times (\nabla \times \vec{A}) &= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \\ \nabla \cdot (\psi \vec{A}) &= \vec{A} \cdot \nabla \psi + \psi \nabla \cdot \vec{A} \\ \nabla \times (\psi \vec{A}) &= \nabla \psi \times \vec{A} + \psi \nabla \times \vec{A} \\ \nabla(\vec{A} \cdot \vec{B}) &= (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) \\ \nabla \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \\ \nabla \times (\vec{A} \times \vec{B}) &= \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}\end{aligned}$$

Differential-Operatoren

Kartesische Koordinaten (x_1, x_2, x_3) :

$$\begin{aligned}\nabla \psi &= \vec{e}_1 \frac{\partial \psi}{\partial x_1} + \vec{e}_2 \frac{\partial \psi}{\partial x_2} + \vec{e}_3 \frac{\partial \psi}{\partial x_3} \\ \nabla \cdot \vec{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\ \nabla \times \vec{A} &= \vec{e}_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \vec{e}_2 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \vec{e}_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) \\ \nabla^2 \psi &= \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2}\end{aligned}$$

Zylinderkoordinaten (r, ϕ, z) :

$$\begin{aligned}\nabla \psi &= \vec{e}_r \frac{\partial \psi}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial \psi}{\partial \phi} + \vec{e}_z \frac{\partial \psi}{\partial z} \\ \nabla \cdot \vec{A} &= \frac{1}{r} \frac{\partial(r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\ \nabla \times \vec{A} &= \vec{e}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \vec{e}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \vec{e}_z \frac{1}{r} \left(\frac{\partial(r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \\ \nabla^2 \psi &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}\end{aligned}$$

Kugelkoordinaten (r, θ, ϕ) :

$$\begin{aligned}\nabla\psi &= \vec{e}_r \frac{\partial\psi}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial\psi}{\partial\theta} + \vec{e}_\phi \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial\phi} \\ \nabla \cdot \vec{A} &= \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial\phi} \\ \nabla \times \vec{A} &= \vec{e}_r \frac{1}{r \sin\theta} \left[\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi} \right] + \vec{e}_\theta \left[\frac{1}{r \sin\theta} \frac{\partial A_r}{\partial\phi} - \frac{1}{r} \frac{\partial(r A_\phi)}{\partial r} \right] + \vec{e}_\phi \frac{1}{r} \left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta} \right] \\ \nabla^2\psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \\ &\equiv \frac{1}{r} \frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2}\end{aligned}$$

Kugelflächenfunktionen

Das Koordinatensystem:

$$\begin{aligned}x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta\end{aligned}$$

Die $Y_{l,m}$ für $l = 0, 1, 2$ sind:

$$\begin{aligned}Y_{0,0} &= \frac{1}{\sqrt{4\pi}} \\ Y_{1,0} &= \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y_{1,1} &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_{1,-1} &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \\ Y_{2,0} &= \sqrt{\frac{5}{4\pi}} \frac{1}{2} (3 \cos^2\theta - 1) \\ Y_{2,1} &= -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \\ Y_{2,-1} &= \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{-i\phi} \\ Y_{2,2} &= \sqrt{\frac{15}{2\pi}} \frac{1}{4} \sin^2\theta e^{2i\phi} \\ Y_{2,-2} &= \sqrt{\frac{15}{2\pi}} \frac{1}{4} \sin^2\theta e^{-2i\phi}\end{aligned}$$

Die assoziierten Legendre-Polynome für $l = 0, 1, 2$ sind:

$$\begin{aligned}P_0^0(\cos \theta) &= 1 \\P_1^0(\cos \theta) &= \cos \theta \\P_1^1(\cos \theta) &= -\sqrt{1 - \cos^2 \theta} \\P_1^{-1}(\cos \theta) &= \frac{1}{2} \sqrt{1 - \cos^2 \theta} \\P_2^0(\cos \theta) &= \frac{1}{2}(3 \cos^2 \theta - 1) \\P_2^1(\cos \theta) &= -3 \sqrt{1 - \cos^2 \theta} \cos \theta \\P_2^{-1}(\cos \theta) &= \frac{1}{2} \sqrt{1 - \cos^2 \theta} \cos \theta \\P_2^2(\cos \theta) &= 3(1 - \cos^2 \theta) \\P_2^{-2}(\cos \theta) &= \frac{1}{8}(1 - \cos^2 \theta)\end{aligned}$$