05.12.2001 Übungen zur Linearen Algebra I

Blatt 8

Aufgabe 1

a) $v_1 = (1,2,4), v_2 = (1,0,-1), v_3 = (0,1,2)$

(α)

$$\begin{aligned} 1x + 1y + 0z &= 0 \\ 2x + 0y + 1z &= 0 \\ 4x - 1y + 2z &= 0 \end{aligned}$$

$$\begin{aligned} x + y &= 0 \\ 2x + z &= 0 \\ 4x - y + 2z &= 0 \end{aligned} \quad \oplus$$

$$\begin{aligned} 2x + z &= 0 \quad | \cdot (-2) \\ 5x + 2z &= 0 \end{aligned}$$

$$\begin{aligned} -4x - 2z &= 0 \\ 5x + 2z &= 0 \end{aligned} \quad \oplus$$

$$\begin{aligned} x &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}$$

v_1, v_2, v_3 l.u. im $\mathbb{R}^3$ Basis, wegen $\dim(\mathrm{Lin}(v_1, v_2, v_3)) = 3$ und $\dim(\mathbb{R}^3) = 3$

$$\begin{aligned} x + y &= 1 \\ 2x + z &= 1 \\ 4x - y + 2z &= 3 \end{aligned} \quad \oplus$$

$$\begin{aligned} 2x + z &= 1 \quad | \cdot (-2) \\ 5x + 2z &= 4 \end{aligned}$$

$$\begin{aligned} -4x - 2z &= -2 \\ 5x + 2z &= 4 \end{aligned} \quad \oplus$$

$$\begin{aligned} x &= 2 \\ y &= -1 \\ z &= -3 \end{aligned}$$

$\Rightarrow \underline{(1,1,3) = 2 \cdot (1,2,4) - (1,0,-1) - 3(0,1,2)}$

b) $(v_1, \dots, v_3)$ $\quad (\mathbb{R}^3)^* = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R})$

$(v_1^*, \dots, v_3^*)$ Basis $(\mathbb{R}^3)^*$

$v_1 = (1,2,4)$

$v_2 = (1,0,-1)$

$v_3 = (0,1,2)$

Duale Basis definiert durch $v_i^*(v_j) = \delta_{ij}$

$v_i \in (\mathbb{R}^3)^*$ $\quad v_i^* : \mathbb{R}^3 \underset{\text{linear}}{\to} \mathbb{R}; \ (x,y,z) \mapsto v_i^*(x,y,z)$

$\varphi : \mathbb{R}^3 \underset{\text{linear}}{\to} \mathbb{R}; \ (x,y,z) \mapsto -2x - 3y + 4z$

$v_1^*(x,y,z) = \alpha_1 x + \beta_1 y + \gamma_1 z$

$v_2^*(x,y,z) = \alpha_2 x + \beta_2 y + \gamma_2 z$

$v_3^*(x,y,z) = \alpha_3 x + \beta_3 y + \gamma_3 z$

$v_1^*(v_1) = 1 \quad v_1^*(v_2) = 0 \quad v_1^*(v_3) = 0$

$v_2^*(v_1) = 0 \quad v_2^*(v_2) = 1 \quad v_2^*(v_3) = 0$

$v_3^*(v_1) = 0 \quad v_3^*(v_2) = 0 \quad v_3^*(v_3) = 1$

$v_1^*(v_1): \ \alpha_1 + 2\beta_1 + 4\gamma_1 = 1$

$v_1^*(v_2): \ \alpha_1 - \gamma_1 = 0$

$v_1^*(v_3): \ \beta_1 + 2\gamma_1 = 0$

$$\boxed{\alpha_1 = 1, \quad \beta_1 = -2, \quad \gamma_1 = 1}$$

$v_2^*(v_1): \ \alpha_2 + 2\beta_2 + 4\gamma_1 = 0$

$v_2^*(v_2): \ \alpha_2 - \gamma_2 = 1$

$v_2^*(v_3): \ \beta_2 + 2\gamma_2 = 0$

$2\beta_2 + 5\gamma_2 = -1$

$\beta_2 + 2\gamma_2 = 0 \ | \cdot (-2)$

$2\beta_2 + 5\gamma_2 = -1$

$-2\beta_2 - 4\gamma_2 = 0$

$$\boxed{\gamma_2 = -1, \quad \beta_2 = 2, \quad \alpha_2 = 0}$$

$v_3^*(v_1): \ \alpha_3 + 2\beta_3 + 4\gamma_3 = 0$

$v_3^*(v_2): \ \alpha_3 - \gamma_3 = 0$

$v_3^*(v_3): \ \beta_3 + 2\gamma_3 = 1$

$$\boxed{\alpha_3 = -2, \quad \beta_3 = 5, \quad \gamma_3 = -2}$$

$a(x - 2y + z) + b(0x + 2y - z) + c(-2x + 5y - 2z) = -2x - 3y + 4z$

$x(a - 2c) + y \cdot (-2a + 2b + 5y) + z(a - b - 2z) = -2x - 3y + 4z$

$$\left.\begin{aligned} (a - 2c) &= -2 \\ -2a + 2b + 5\cancel{c} &= -3 \\ a - b - 2c &= 4 \end{aligned}\right\} \Rightarrow \begin{cases} a = -\frac{8}{3} \\ b = -\frac{10}{3} \\ c = -\frac{1}{3} \end{cases}$$

Linearkombination:

$a v_1^* + b v_2^* + c v_3^* = \varphi$

06.12.2011 Aufgabe 2

$u_1 = (1,3)$, $u_2 = (2,1)$, $u_3 = (4,7)$

(a)

$$\frac{2\cdot(2,1) - (4,7)}{-5} = (0,1)$$

$$\frac{(1,3) - 3\cdot(2,1)}{-5} = (1,0) \quad ✓$$

$\Rightarrow$ Erzeugenden System $\square$

(b)

$$\left.\begin{array}{l} f(u_1) = (-2,-1) \\ f(u_2) = (6,3) \\ f(u_3) = (2,1) \end{array}\right\} \xrightarrow{\text{gilt}} f: \mathbb{R}^2 \to \mathbb{R}^2 ;\ (x,y) \mapsto (x', y')$$

Ansatz: $x' = \alpha x + \beta y$

$y' = \gamma x + \delta y$

$-2 = \alpha + 3\beta \Leftrightarrow \alpha = -2-3\beta \qquad -1 = \gamma + 3\delta \qquad \Rightarrow -1 = \gamma + 9 - 6\gamma \Leftrightarrow -10 = -5\gamma$

$6 = 2\alpha + \beta \qquad 3 = 2\gamma + \delta \Rightarrow \delta = 3 - 2\gamma \qquad \gamma = 2$

$\Rightarrow \delta = -1$

$2 = 4\alpha + 7\beta \qquad 1 = 4\gamma + 7\delta$

$\Rightarrow 6 = -4 - 6\beta + \beta \qquad \gamma = 2$

$10 = -5\beta \qquad \delta = -1$

$\beta = -2$

$\alpha = 4$

$$f = (4x - 2y,\ 2x - y)$$

c)

(d) $im(f) = \{(4x-2y, 2x-y) \mid x,y \in \mathbb{R}\}$

$\forall x,y : \underbrace{(2\alpha - \alpha)}_{\alpha(2;1)}$; $\alpha = 2x-y$ $\forall \alpha \in \mathbb{R} \Rightarrow im(f) = lin((2,1))$ ✓

$ker(f) = \{(x,y) \mid f(x,y) = 0\}$

$= \{(x,y) \mid 2x = y\}$ $4x-2y = 0 \wedge 2x-y=0$

$= \{(x, 2x) \mid x \in \mathbb{R}\}$ $\Leftrightarrow 2x = y$

$= \{x(1,2) \mid x \in \mathbb{R}\}$ $4x-4x=0$

$= Lin((1,2))$ $\underline{\underline{0=0}} \wedge \underline{\underline{2x=y}}$

$\Rightarrow b = (1,2)$

$dim(im(f)) = Rg(f) = 1$