differential geometry

- equivalence principle → weird!

gravitational forces can be transformed away by moving to a set of <u>local</u> inertial coordinates

in the same way, inertial forces can be made to disappear

→ gravity is an inertial force

→ both gravity and inertial forces appear in noninertial coords

but if both are equivalent

→ can I write every inertial force as a gravitational force

→ and if yes, which matter distribution does it source?

Newton equation of motion in an accelerated frame

$$\frac{d^2\vec{x}}{dt^2} = \underbrace{-\nabla\phi}_{\text{potential}} \left(- \dot{\vec{\omega}}\times\vec{x}\right) \underbrace{- \vec{\omega}\times(\vec{\omega}\times\vec{x})}_{\text{centrifugal}} \underbrace{- 2(\omega\times\dot{\vec{x}})}_{\text{Coriolis}}$$

- $\phi_z = \frac{1}{2c}|\vec{\omega}\times\vec{x}|^2 = \frac{1}{2c} A_z^2$

$$\vec{A}_z = \vec{\omega}\times\vec{x}$$

Looks a bit like a gravitational Lorentz-force

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\phi\left(-\frac{1}{c}\frac{\partial A_z}{\partial t}\right) - \nabla\phi_z - \frac{1}{c}\operatorname{rot}\vec{A}_z\times\dot{\vec{x}}$$

and this would exactly follow from $\Gamma^{\alpha}_{\mu\nu}$ for $g_{00} = \left(1 - \frac{2}{c^2}\phi\right)$ at order $\frac{v}{c}$ in the geodesic eqn

→ introducing gravity as a 4-force would be ok at the approximation β

→ but $m_i = m_g$ would be a mystery

- are ϕ_z and $\vec{A}_z$ actual grav. fields?

→ Mach's principle: yes!

differential geometry

○ weak Newtonian fields

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$ not only locally!

($\eta_{\mu\nu}$: Minkowski, $h_{\mu\nu}$: weak perturbation)

+ assume that metric perturbation is stationary $\partial_0 h_{\mu\nu} = 0$

geodesic equation $\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$

slow moving: $\left|\frac{dx^i}{d\tau}\right| \ll \left|\frac{dx^0}{d\tau}\right|$ → 0-component dominates

Christoffel-symbol $\Gamma^\rho_{00} = \frac{g^{\rho\sigma}}{2}(\partial_0 g_{0\sigma} + \partial_0 g_{\sigma 0} - \partial_\sigma g_{00})$

$$\stackrel{!}{=} -\frac{1}{2}\eta^{\rho\sigma}\partial_\sigma h_{00}$$

with $\Gamma^0_{00} = 0$, $\Gamma^i_{00} = \frac{1}{2}\delta^{ij}\partial_j h_{00}$

→ $\frac{d^2 x^0}{d\tau^2} = \frac{d^2 t}{d\tau^2} = 0$

$\frac{d^2 x^i}{d\tau^2} = -\frac{c^2}{2} \cdot \left(\frac{dt}{d\tau}\right)^2 \cdot \nabla h_{00}$

} $\frac{d^2 \vec{x}}{dt^2} = -\frac{c^2}{2}\nabla h_{00}$

→ identify h_{00} with $2 \cdot \frac{\phi}{c^2}$

→ $g_{00} = \left(1 + \frac{2\phi}{c^2}\right)$

alternative interpretation: $\tau(t)$ is affected by gravitational fields → $c^2 d\tau^2 = g_{00} c^2 dt^2$

→ time dilation in grav. fields $1 + \frac{2\phi}{c^2} < 1$, $\phi < 0$

• 'rubber sheet'-analogy shows the wrong thing:

the 'curvature' in dt is responsible for e.g. the motion of planets differential geometry : tensors

- motion in a gravitational field is described by the connection Γ, which contains derivatives of the metric → purely geometric picture of motion → Γ is the gravitational field.

- metric $g_{\mu\nu}$ should follow from the matter distribution in a 2nd order differential equation:

$$\mathcal{L} = \frac{(\nabla\phi)^2}{2} - 4\pi G \rho\phi \rightarrow \nabla \frac{\partial\mathcal{L}}{\partial\nabla\phi} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \text{EL-eqn}$$

Lagrange density

$$\Delta\phi - 4\pi G\rho = 0 \quad \text{Poisson}$$

but this relation should be
(1) a tensor-relation for $g_{\mu\nu}$
(2) include energy-momentum$_{\mu\nu}$
(3) geometric

- tensors ~ linear forms which map vectors onto real numbers
  - rank n: mapping of n vectors onto $\mathbb{R}$.
  - linearity: $\bar{t}(\alpha\vec{u} + \beta\vec{v}) = \alpha\,\bar{t}(\vec{u}) + \beta\,\bar{t}(\vec{v})$ for rank 1.
  - components: use the basis $\begin{cases} \bar{t}(\vec{e}_\mu) = t_\mu & \text{covariant} \\ \bar{t}(\vec{e}^\mu) = t^\mu & \text{contravariant} \end{cases}$

$$\bar{t}(\vec{e}_\mu, \vec{e}_\nu) = t_{\mu\nu}$$
$$\bar{t}(\vec{e}_\mu, \vec{e}^\nu) = t_\mu{}^\nu \quad \text{for rank 2}$$
$$\bar{t}(\vec{e}^\mu, \vec{e}^\nu) = t^{\mu\nu}$$

  - assemble vectors from their basis

$$\bar{t}(\vec{u}, \vec{v}) = \bar{t}(u^\mu\vec{e}_\mu, v^\nu\vec{e}_\nu) = u^\mu \cdot v^\nu\, \bar{t}(\vec{e}_\mu, \vec{e}_\nu) = u^\mu v^\nu t_{\mu\nu}$$

↓

$u_\mu\vec{e}^\mu$ exchange possible, leading to $u_\mu v^\nu t^\mu{}_\nu$ etc.

  - symmetry: $\bar{t}(\vec{u}, \vec{v}) = \pm\,\bar{t}(\vec{v}, \vec{u})$ symmetric or antisymmetric

differential geometry

○ metric tensor

$$g(\vec{u},\vec{v}) = \vec{u}\cdot\vec{v} = g_{\mu\nu} u^\mu v^\nu$$

with components $g_{\mu\nu} = g(\vec{e}_\mu,\vec{e}_\nu) = \vec{e}_\mu \vec{e}_\nu$

and inverse $g^{\mu\nu} = g(\vec{e}^\mu,\vec{e}^\nu) = \vec{e}^\mu\cdot\vec{e}^\nu$

and reciprocity $g(\vec{e}^\mu,\vec{e}_\nu) = g(\vec{e}_\mu;\vec{e}^\nu) = \delta^\mu{}_\nu$

○ operations on tensors

- addition $(\bar{t}\pm\bar{s})(\vec{e}_\mu,\vec{e}_\nu) = \bar{t}(\vec{e}_\mu,\vec{e}_\nu)\pm\bar{s}(\vec{e}_\mu,\vec{e}_\nu) = t_{\mu\nu}\pm s_{\mu\nu}$

- scaling $(\alpha\bar{t})(\vec{e}_\mu,\vec{e}_\nu) = \alpha\bar{t}(\vec{e}_\mu,\vec{e}_\nu) = \alpha t_{\mu\nu}$

- contractions $\bar{t}(\vec{e}_\mu,\vec{e}^\mu) = t_\mu{}^\mu$, or across two tensors

  $\bar{t}(\vec{e}_\mu,\vec{e}_\nu)\,\bar{s}(\vec{e}^\nu,\vec{e}^\rho) = t_{\mu\nu}s^{\nu\rho}$

  $\longrightarrow$ always reduces rank by 2

- outer product

  2 vectors $(\vec{u}\otimes\vec{v})(\vec{e}_\mu,\vec{e}_\nu) = \vec{u}(\vec{e}_\mu)\vec{v}(\vec{e}_\nu) = u_\mu v_\nu$

  in general $\vec{u}\otimes\vec{v} \neq \vec{v}\otimes\vec{u}$

  multiplication of tensor components in given basis "table of all products"

- relation to vectors: rank 1 tensors

  $\bar{t}(\vec{e}_\mu) = t_\mu = \vec{t}\,\vec{e}_\mu$, $\bar{t}(\vec{e}^\mu) = t^\mu = \vec{t}\cdot\vec{e}^\mu$

- invert the projection ~ reassemble $\bar{t}$ from $t_{\mu\nu}$:

  (1) $(\vec{e}_\mu\otimes\vec{e}_\nu)(\vec{e}^\rho,\vec{e}^\sigma) = \vec{e}_\mu(\vec{e}^\rho)\cdot\vec{e}_\nu(\vec{e}^\sigma) = \delta_\mu{}^\rho\cdot\delta_\nu{}^\sigma$

  (2) multiply with $t^{\mu\nu}$

  $t^{\mu\nu}\cdot(\vec{e}_\mu\otimes\vec{e}_\nu)(\vec{e}^\rho,\vec{e}^\sigma) = t^{\mu\nu}\delta_\mu{}^\rho\delta_\nu{}^\sigma = t^{\rho\sigma}$ ~ correct!

  (3) set $\bar{t} = t^{\mu\nu}(\vec{e}_\mu\otimes\vec{e}_\nu)$

  $\longrightarrow$ $\bar{t}$ reconstructed from $t^{\mu\nu}$, projection with $\vec{e}_\mu,\vec{e}_\nu$ yields components again.

differential geometry

○ behaviour under coordinate transforms

coordinate transform $x^\mu \to x'^\mu(x^\nu)$

causes a change in basis vector $\vec{e}'_\mu = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{\text{Jacobian}} \vec{e}_\nu$

and the dual $\vec{e}'^\mu = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}}_{\text{inverse Jacobian}} \vec{e}^\nu$

for tensors: $t'_{\mu\nu} = \bar{t}(\vec{e}'_\mu, \vec{e}'_\nu) = \bar{t}\left(\frac{\partial x^\rho}{\partial x'^\mu}\vec{e}_\rho, \frac{\partial x^\sigma}{\partial x'^\nu}\vec{e}_\sigma\right)$

$$= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\,\bar{t}(\vec{e}_\rho\,\vec{e}_\sigma) = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\,t_{\rho\sigma}$$

↑ linearity.

it's important to remember that $t'_{\mu\nu}$ and $t_{\mu\nu}$ are the components of the <u>same</u> tensor in <u>different</u> bases

$$\bar{t} = t'_{\mu\nu}\left(\vec{e}'^\mu \otimes \vec{e}'^\nu\right) = t_{\mu\nu}\left(\vec{e}^\mu \otimes \vec{e}^\nu\right).$$

similarly $t'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\, t^{\rho\sigma}$

tensor-relations are geometric statements about tensors and <u>hold</u> component by component in every coordinate frame

in particular, if you can show that a tensor $= 0$ in one (possibly constructed) frame, it's zero in all frames.

○ covariant derivative → in analogy to the covariant derivative of a vector

· for vectors

$$\partial_\rho \vec{v} = \partial_\rho(v^\mu \vec{e}_\mu) = (\partial_\rho v^\mu)\vec{e}_\mu + v^\mu \underline{\partial_\rho \vec{e}_\mu}$$
$$= (\partial_\rho v^\mu)\cdot\vec{e}_\mu + v^\mu \Gamma^\sigma_{\mu\rho} e_\sigma$$

· for tensors, rank 2

$$\partial_\rho \bar{t} = \partial_\rho(t^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu)$$
$$= \partial_\rho t^{\mu\nu}\cdot\vec{e}_\mu\otimes\vec{e}_\nu + t^{\mu\nu}\cdot(\partial_\rho\vec{e}_\mu)\otimes\vec{e}_\nu + t^{\mu\nu}\vec{e}_\mu\otimes\partial_\rho\vec{e}_\nu$$
$$= \partial_\rho t^{\mu\nu}\cdot\vec{e}_\mu\otimes\vec{e}_\nu + t^{\mu\nu}\Gamma^\sigma_{\mu\rho}\vec{e}_\sigma\otimes\vec{e}_\nu + t^{\mu\nu}\Gamma^\sigma_{\nu\rho}\vec{e}_\mu\otimes\vec{e}_\sigma$$
$$= \left(\partial_\rho t^{\mu\nu} + \Gamma^\mu_{\sigma\rho}t^{\sigma\nu} + \Gamma^\nu_{\sigma\rho}t^{\mu\sigma}\right)\vec{e}_\mu\otimes\vec{e}_\nu$$
$$= \nabla_c\, t^{\mu\nu}\quad \vec{e}_\mu\otimes\vec{e}_\nu$$

○ differential geometry

analogous formulae apply to covariant or mixed comps

→ watch out for sign change!

○ covariant derivative of the metric $\nabla g = 0$

$$\nabla_\xi g_{\mu\nu} = \partial_\xi g_{\mu\nu} - \Gamma^\sigma_{\mu\xi} g_{\sigma\nu} - \Gamma^\sigma_{\nu\xi} g_{\mu\sigma}$$
$$= \Gamma^\sigma_{\mu\xi} g_{\sigma\nu} + \Gamma^\sigma_{\nu\xi} g_{\mu\sigma} - \Gamma^\sigma_{\mu\xi} g_{\sigma\nu} - \Gamma^\sigma_{\nu\xi} g_{\mu\nu}$$
$$= 0$$

→ raising indices with $g^{\mu\nu}$ or lowering indices with $g_{\mu\nu}$

$$\nabla_\xi t^{\mu\nu} = \nabla_\xi (g^{\mu\sigma} t_\sigma{}^\nu) = \underbrace{\nabla_\xi g^{\mu\sigma} \cdot t_\sigma{}^\nu}_{=0} + g^{\mu\sigma} \cdot \nabla_\xi t_\sigma{}^\nu$$

○ derivative of tensors along curves: $x^\sigma(\lambda)$

$$\bar{t}(\lambda) = t^{\mu\nu}(\lambda) \cdot \vec{e}_\mu(\lambda) \otimes \vec{e}_\nu(\lambda)$$

in analogy to the derivative $\frac{d\vec{v}}{d\lambda}$:

$$\frac{d\bar{t}}{d\lambda} = \frac{dt^{\mu\nu}}{d\lambda} \vec{e}_\mu \otimes \vec{e}_\nu + t^{\mu\nu} \frac{d\vec{e}_\mu}{d\lambda} \otimes \vec{e}_\nu + t^{\mu\nu} \vec{e}_\mu \otimes \frac{d\vec{e}_\nu}{d\lambda}$$

$$= \frac{dt^{\mu\nu}}{d\lambda} \vec{e}_\mu \otimes \vec{e}_\nu + t^{\mu\nu} \frac{dx^\xi}{d\lambda} \underbrace{\frac{\partial \vec{e}_\mu}{\partial x^\xi}} \otimes \vec{e}_\nu + t^{\mu\nu} \vec{e}_\mu \otimes \frac{dx^\xi}{d\lambda} \underbrace{\frac{\partial \vec{e}_\nu}{\partial x^\xi}}$$

$$= \left(\frac{dt^{\mu\nu}}{d\lambda} + \Gamma^\mu{}_{\sigma\xi} t^{\sigma\nu} \cdot \frac{dx^\xi}{d\lambda} + \Gamma^\nu{}_{\sigma\xi} \cdot t^{\mu\sigma} \frac{dx^\xi}{d\lambda} \right) \vec{e}_\mu \otimes \vec{e}_\nu$$

$$= \frac{Dt^{\mu\nu}}{D\lambda} \vec{e}_\mu \otimes \vec{e}_\nu$$

○ curvature

- equivalence means: you can find a coordinate transform describing an accelerated frame of reference
  → in this frame, the metric is $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$
- but some set of coordinates $ds^2 = g_{\mu\nu} dx'^\mu dx'^\nu$ might already describe Minkowski-space, but the bad choice of coordinates does not reveal it.