differential geometry

- derivative of a vector along a curve

$\vec{v}(\lambda) = v^\mu(\lambda)\,\vec{e}_\mu(\lambda)$ with a parameter λ, coordinates $x^\sigma(\lambda)$

$$\rightarrow \frac{d\vec{v}}{d\lambda} = \frac{dv^\mu}{d\lambda}\vec{e}_\mu + v^\mu \frac{d\vec{e}_\mu}{d\lambda} = \frac{dv^\mu}{d\lambda} + v^\mu \cdot \frac{dx^\sigma}{d\lambda}\underline{\frac{\partial \vec{e}_\mu}{\partial x^\sigma}}$$

$$= \frac{dv^\mu}{d\lambda} + \Gamma^\rho_{\mu\sigma}\, v^\mu \frac{dx^\sigma}{d\lambda}\vec{e}_\rho \qquad \mu \rightleftarrows \rho$$

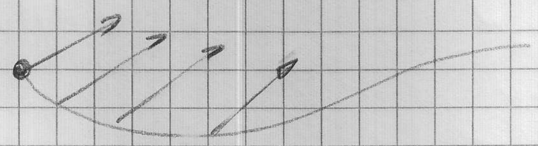
$$= \left[\frac{dv^\mu}{d\lambda} + \Gamma^\mu{}_{\rho\sigma} v^\rho \frac{dx^\sigma}{d\lambda}\right]\cdot \vec{e}_\mu$$

intrinsic derivative ~ analogy to $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\nabla$

- parallel transport

shift vector along curve with parameter λ such that

$\frac{d\vec{v}}{d\lambda} = 0 \longrightarrow \vec{v}$ is parallel to initial $\vec{v}$ for all λ



$$\frac{d\vec{v}}{d\lambda} = 0 \longrightarrow \frac{dv^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} v^\rho \frac{\partial x^\sigma}{\partial\lambda} = 0$$

$$\longrightarrow v^\mu(\lambda) = -\int_{\lambda_i}^{\lambda} d\lambda \quad \Gamma^\mu{}_{\rho\sigma} v^\rho \frac{dx^\sigma}{d\lambda}$$

- determined by initial direction $v^\mu(\lambda_i)$
- but is path dependent due to varying $\Gamma^\mu{}_{\rho\sigma}$
- $\vec{v}$ not equal to $\vec{v}(\lambda_i)$ after transport through a closed loop

- affine parameters

2 types $\nearrow$ non-null ~ massive particles $ds^2 > 0$

2 types $\searrow$ null ~ photons $ds^2 = 0$

curve $x^\sigma(\lambda) \longrightarrow$ tangent at A $\vec{t} = \frac{dx^\sigma}{d\lambda}\cdot\vec{e}_\sigma$

with length $|\vec{t}| = \sqrt{g_{\mu\nu} t^\mu t^\nu} = \frac{\sqrt{g_{\mu\nu} dx^\mu dx^\nu}}{d\lambda} = \left|\frac{ds}{d\lambda}\right|$

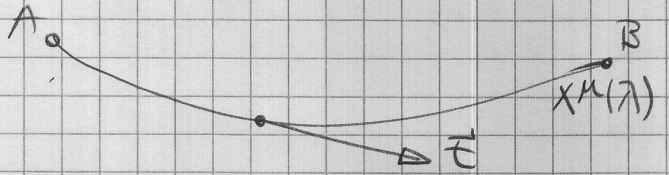
differential geometry

- affine parameters, cont'ed

$\frac{|ds|}{d\lambda} = |\vec{t}| \rightarrow$ affine parametersation $\lambda = as + b$ gives a constant length of the tangent.

but: if $ds = 0$ (as for photons) any parameter could be used but 'length' is meaningless

- geodesics (only torsion-free manifolds)
  - 'shortest' connection between two points $\rightarrow$ definition for non-null curves
  - curves with constant tangents $\rightarrow$ definition for non-null **or** null curves.



fixed tangent: $\frac{d\vec{t}}{d\lambda} = \underbrace{c(\lambda)}_{\text{proportionality}} \vec{t}$

$\rightarrow \frac{Dt^\mu}{D\lambda} = \frac{dt^\mu}{d\lambda} + \Gamma^\mu{}_{\rho\sigma} t^\rho \frac{dx^\sigma}{d\lambda} = c(\lambda)\, t^\mu$

$\rightarrow t^\mu = \frac{dx^\mu}{d\lambda}$ yields the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \underbrace{c(\lambda)}\cdot\frac{dx^\mu}{d\lambda}$$

works for null curves and non-null curves

- geodesic: $\vec{t}$ always points into the same direction

$\vec{t} = \text{const} \rightarrow \frac{d\vec{t}}{d\lambda} = 0 \rightarrow c(\lambda) = 0 \rightarrow \frac{Dt^\mu}{d\lambda} = 0$

- $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$ for affinely parametrised curves with constant tangent direction

geodesic equation

differential geometry

o reparameterise a geodesic $\lambda \to \lambda'$

$$\frac{d^2 x^\mu}{d\lambda'^2} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{d\lambda'} \frac{dx^\sigma}{d\lambda'} = 0$$

$$= \frac{d}{d\lambda'}\left(\frac{d\lambda}{d\lambda'}\frac{d}{d\lambda} x^\mu\right) + \Gamma^\mu{}_{\rho\sigma} \frac{d\lambda}{d\lambda'}\frac{dx^\rho}{d\lambda}\frac{d\lambda}{d\lambda'}\frac{dx^\sigma}{d\lambda}$$

$$= \frac{d^2\lambda}{d\lambda'^2}\frac{dx^\mu}{d\lambda} + \left(\frac{d\lambda}{d\lambda'}\right)^2 \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma} \frac{d\lambda}{d\lambda'}\frac{dx^\rho}{d\lambda}\frac{d\lambda}{d\lambda'}\frac{dx^\sigma}{d\lambda}$$

$$\to \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma} \cdot \frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda^2} = \underbrace{- \frac{dx^\mu}{d\lambda} \cdot \left(\frac{d\lambda'}{d\lambda}\right)^2 \cdot \frac{d^2\lambda}{d\lambda'^2}}_{\text{new term}}$$

but. if λ is affine, then any linear transform λ' is affine as well

$$\lambda = a\lambda' + b \to \frac{d^2\lambda}{d\lambda'^2} = 0$$ — last term vanishes + geodesic eqn fulfilled

o 2nd definition: shortest connection in given metric

minimise $\int_A^B ds = \int_A^B \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$

in analogy to variation of the Lagrange function

$\delta \int_A^B ds = \delta \int_A^B L\, d\lambda$: use Euler-Lagrange eqns, $\dot{s} = \frac{ds}{d\lambda} = L$

$$\longrightarrow \frac{d}{d\lambda}\frac{\partial L}{\partial\left(\frac{dx^\sigma}{d\lambda}\right)} - \frac{\partial L}{\partial x^\sigma} = 0$$

this gives:

$$\frac{d}{d\lambda}\frac{1}{2\dot{s}}\left(g_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\cdot\delta^\nu{}_\sigma + \frac{dx^\nu}{d\lambda}\cdot\delta^\mu{}_\sigma\right)\right) = \frac{1}{2\dot{s}}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

$$\to \underline{\frac{d}{d\lambda}\frac{1}{\dot{s}}\left(g_{\nu\sigma}\frac{dx^\nu}{d\lambda}\right)} = \frac{1}{2\dot{s}}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

$$= \frac{1}{\dot{s}}\left(\frac{\partial}{\partial x^\nu} g_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2} - \frac{\ddot{s}}{\dot{s}} g_{\mu\sigma}\frac{dx^\mu}{d\lambda}\right)$$

with $\frac{d}{d\lambda} g_{\mu\sigma} = \frac{\partial}{\partial x^\nu} g_{\mu\sigma} \cdot \frac{dx^\nu}{d\lambda}$

differential geometry

○ geodesic equation

$$g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2} + \frac{\partial}{\partial x^\nu}g_{\mu\sigma}\cdot\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}\frac{\partial}{\partial x^\sigma}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{\ddot{s}}{\dot{s}}g_{\mu\sigma}\frac{dx^\mu}{d\lambda}$$

$$\rightarrow \frac{d^2x^\varrho}{d\lambda^2} + \frac{1}{2}g^{\varrho\sigma}\left[\frac{\partial}{\partial x^\nu}g_{\mu\sigma} + \frac{\partial}{\partial x^\mu}g_{\nu\sigma} - \frac{\partial}{\partial x^\sigma}g_{\mu\nu}\right]\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

if • $\frac{\ddot{s}}{\dot{s}} = 0$ by affine parameterisation

• symmetrisation $\frac{\partial}{\partial x^\nu}g_{\mu\sigma} = \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu}\right)$

• identity Christoffel symbol $\Gamma^\varrho{}_{\mu\nu}$

$$\rightarrow \frac{d^2x^\varrho}{d\lambda^2} + \Gamma^\varrho{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

'cyclic' coordinates: metric $g_{\mu\nu}$ does not depend on x^ϱ

$$\rightarrow \frac{\partial L}{\partial \dot{x}^\varrho} = g_{\varrho\nu}\underbrace{\frac{dx^\nu}{d\lambda}}_{\text{tangent vector}} = \text{const}$$

$\rightarrow$ t^ϱ is constant along a geodesic (Killing!)

○ alternative form of the geodesic equation

geodesic: tangent is constant, $\frac{d\vec{t}}{d\lambda} = 0$

$$\rightarrow \frac{Dt_\varrho}{D\lambda} = \frac{dt_\varrho}{d\lambda} - \Gamma^\nu{}_{\mu\varrho}\cdot t_\nu\cdot\underbrace{\frac{dx^\mu}{d\lambda}}_{t^\mu} = 0$$

$$\rightarrow \dot{t}_\varrho = \Gamma^\nu{}_{\mu\varrho}\, t_\nu t^\mu$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^\varrho}g_{\sigma\mu} + \underbrace{\frac{\partial}{\partial x^\mu}g_{\varrho\sigma} - \frac{\partial}{\partial x^\sigma}g_{\varrho\mu}}_{=0}\right)t^\mu t^\sigma$$

$$= \frac{1}{2}\frac{\partial}{\partial x^\varrho}g_{\sigma\mu}t^\mu t^\sigma$$

if $g_{\sigma\mu}$ does not depend on x^ϱ $\rightarrow$ $\dot{t}_\varrho = 0$, $t_\varrho = \text{const.}$

differential geometry

○ apply to the principle of least proper time:

$$\int ds = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau$$

would yield an equation of motion

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0$$

if $g_{\mu\nu} = \eta_{\mu\nu}$ (back to Minkowski-geometry),

$$\frac{\partial \vec{e}_\mu}{dx^\rho} = 0 \rightarrow \Gamma^\rho_{\mu\nu} = 0 \rightarrow \frac{d^2 x^\mu}{d\tau^2} = 0$$ inertial motion

○ revisit the equivalence principle

- is gravity a 4-force, just like electromagnetism?

perhaps yes: introduce $\frac{q}{c} A_\mu dx^\mu$ into proper time differential $\longrightarrow$ new phenomena, such as $\Delta \vec{A} = \frac{4\pi}{c} \vec{J}$

$\longrightarrow$ $\phi \sim \frac{1}{r}$ well known from coulomb fields

- it would be weird if all objects felt the same acceleration

$$\mathcal{L} = \underbrace{-m_i \cdot c^2 \int d\tau}_{\text{inertia}} + \underbrace{\frac{m_g}{c} \cdot A_\mu dx^\mu}_{\text{coupling to grav. field}}$$

but this could be a principle of Nature, with both m_i and m_g being Lorentz-scalars: $m_i = m_g$

- if gravity would be a geometric effect, this question would not occur!

$$\mathcal{L} = -m_i c \int \sqrt{g_{\mu\nu} u^\mu u^\nu}\, d\tau$$

for the Minkowski-geometry we have $g_{\mu\nu} = \eta_{\mu\nu}$ and for small velocities $u^0 = c$ dominates,

$\rightarrow \mathcal{L} \sim -m_i c^2 \int d\tau$

differential geometry

○ equivalence principle; cont'ed:

- there is very, very good experimental evidence that $m_i = m_g$, at the level of 10^{-11}

→ perhaps gravity is geometric ?!

- variation of the Lagrange-density $\mathcal{L} = -mc\int\sqrt{g_{\mu\nu}u^\mu u^\nu}\,dt$

yield the geodesic equation as the equation of motion

→ particles 'fall' along geodesics (if no other force is present)

- possible geometries are pseudo-Riemannian
  - apply a coordinate transformation: $g \to \eta$ in a vicinity of a point, up to terms $O(\partial^2 g)$

    here, the geodesic equation predicts $\frac{d^2x^\mu}{d\tau^2} = 0$

    ↦ force-free motion!
  - local cartesian coordinates become local inertial coords.
  - corrections are of the shape $\partial^2 g$ → tidal forces
  - equivalence is local

    $g \to \eta$ is only locally possible, with $g = \eta$ and $\partial g = 0$

    basis is orthonormal in the sense $\vec{e}_\mu \cdot \vec{e}_\nu = \eta_{\mu\nu}$

    but Lorentz-transforms respect this definition

    → local inertial coordinates are defined up to Lorentz-transforms

○ 'tease' geodesic equation out of Newton's first law:

(1) $m_i \cdot \ddot{\vec{x}} = -m_g \nabla\phi$

(2) $m_i \ddot{\vec{x}} + m_g \nabla\phi = 0$ force-free motion

(3) $\ddot{\vec{x}} + \nabla\phi = 0$ equivalence $m_i = m_g$

(4) $\ddot{\vec{x}} + c^2 \nabla \frac{\phi}{c^2} = 0$ introduce c

(5) $\frac{d^2x^\mu}{d\tau^2} + u^0 u^0 \cdot \partial^\mu \frac{\phi}{c^2} = 0$ 4-velocity + prop. time

(6) $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00} u^0 u^0 = 0$ with $\Gamma^\mu{}_{00} = \partial^\mu \frac{\phi}{c^2}$

differential geometry

○ weak Newtonian fields

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$ not only locally!

Minkowski — weak perturbation

+ assume that metric perturbation is stationary $\partial_0 h_{\mu\nu} = 0$

geodesic equation $\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$

slow moving: $\left|\frac{dx^i}{d\tau}\right| \ll \left|\frac{dx^0}{d\tau}\right| = c \rightarrow$ 0-component dominates

Christoffel-symbol $\Gamma^\rho_{00} = \frac{g^{\rho\sigma}}{2}\left(\partial_0 g_{0\sigma} + \partial_0 g_{\sigma 0} - \partial_\sigma g_{00}\right)$

$\stackrel{!}{=} -\frac{1}{2}\eta^{\rho\sigma}\partial_\sigma h_{00}$

with $\Gamma^0_{00} = 0$, $\Gamma^i_{00} = \frac{1}{2}\delta^{ij}\partial_j h_{00}$

$$\left.\begin{aligned} \rightarrow \quad & \frac{d^2 x^0}{d\tau^2} = \frac{d^2 t}{d\tau^2} = 0 \\ & \frac{d^2 x^i}{d\tau^2} = -\frac{c^2}{2} \cdot \left(\frac{dt}{d\tau}\right)^2 \cdot \nabla h_{00} \end{aligned}\right\} \quad \frac{d^2 \vec{x}}{dt^2} = -\frac{c^2}{2}\nabla h_{00}$$

$\rightarrow$ identify h_{00} with $2 \cdot \frac{\phi}{c^2}$

$\rightarrow$ $g_{00} = \left(1 + \frac{2\phi}{c^2}\right)$

alternative interpretation: $\tau(t)$ is affected by gravitational fields $\rightarrow c^2 d\tau^2 = g_{00} c^2 dt^2$

$\rightarrow$ time dilation in grav. fields $\quad 1 + \frac{2\phi}{c^2} < 1, \ \phi < 0$

differential geometry

- equivalence principle → weird!

gravitational forces can be transformed away by moving to a set of <u>local</u> inertial coordinates

in the same way, inertial forces can be made to disappear

→ gravity is an inertial force

→ both gravity and inertial forces appear in noninertial coords

but if both are equivalent

→ can I write every inertial force as a gravitational force

→ and if yes, which matter distribution does it source?

Newton equation of motion in an accelerated frame

$$\underset{}{\frac{d^2\vec{x}}{dt^2}} = \underbrace{-\nabla\phi}_{\text{potential}} \left(- \dot{\vec{\omega}} \times \vec{x}\right) \underbrace{- \vec{\omega} \times (\vec{\omega} \times \vec{x})}_{\text{centrifugal}} \underbrace{- 2(\omega \times \dot{\vec{x}})}_{\text{Coriolis}}$$

- $$\phi_z = \frac{1}{2c} |\vec{\omega} \times \vec{x}|^2 = \frac{1}{2c} A_z^2$$

$$\vec{A}_z = \vec{\omega} \times \vec{x}$$

Looks a bit like a gravitational Lorentz-force

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\phi \left(- \frac{1}{c}\frac{\partial A_z}{\partial t}\right) \underbrace{- \nabla\phi_z - \frac{1}{c} \operatorname{rot} \vec{A}_z \times \dot{\vec{x}}}$$

and this would exactly follow from $\Gamma^{\alpha}_{\mu\nu}$ for $g_{00} = \left(1 - \frac{2}{c^2}\phi\right)$ at order $\frac{v}{c}$ in the geodesic eqn

→ introducing gravity as a 4-force would be ok at the approximation β

→ but $m_i = m_g$ would be a mystery

- are ϕ_z and $\vec{A}_z$ actual grav. fields?

→ Mach's principle: yes!