general relativity: manifolds + coordinates

- differentiable manifolds → fancy word for "space"
  - manifolds are sets which can be continuously parameterised
  - # of parameters needed → dimension of the manifold
  - parameters = coordinates $(x^1, .., x^n)$

  sometimes one uses overlapping patches with different sets of coordinates to cover the manifold

  - differentiable manifold: definition of a differentiable field at each point of the manifold possible.

- curves and surfaces

  $x^\alpha = x^\alpha(u)$ $\quad \alpha = 1, .., n$ defines a curve

  $x^\alpha = x^\alpha(u^1, .., u^m)$ $\quad \alpha = 1, .., n$ submanifold

  if $m = n-1$: hypersurface

- coordinate transforms

  choice of coordinates is arbitrary: relevant are the geometric and topological properties → invariances!

  coordinate transformation $x'^\alpha = x'^\alpha(x^\beta)$

  if the coordinate transformation is single valued and differentiable →

  transformation through Jacobian $\frac{\partial x'^\alpha}{\partial x^\beta}$

  varies naturally across manifold

  (if not, this would be a global transformation).

  if $\det\left(\frac{\partial x'^\alpha}{\partial x^\beta}\right) \neq 0$ transformation is invertible: $x^\beta(x'^\alpha)$.

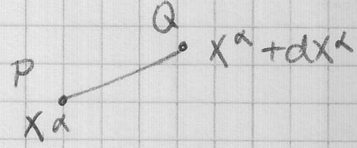
  successive application of $x \to x' \to x$:

  $$\frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\gamma} = \frac{\partial x'^\alpha}{\partial x'^\gamma} = \delta^\alpha{}_\gamma = \begin{cases} 0 & \alpha \neq \gamma \\ 1 & \alpha = \gamma \end{cases}$$

  if the coordinates are independent.

general relativity : manifolds + coordinates

○ coordinate transforms, cont'ed :



$$dx'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} dx^\beta$$ ‹ from individual transforms.

summation convention: automatic summation over 2 indices which appear once as super- and once as subscript.

○ geometry ~ locally varying

invariant distance ds^2 between points: $ds^2 = g(x^\alpha, dx^\alpha)$

position dependent

distance computed locally

Riemannian-geometry $ds^2 = g_{\alpha\beta}(x^\alpha) \cdot dx^\alpha dx^\beta$

as a generalisation to the global Minkowski-geometry

$g_{\mu\nu}$ is symmetric, $g_{\alpha\beta} = g_{\beta\alpha}$, because in the summation with the symmetric $dx^\alpha dx^\beta$ any antisymmetry drops out.

coordinate transform

$$ds^2 = g_{\alpha\beta}(x^\alpha) \cdot dx^\alpha dx^\beta = \underbrace{g_{\alpha\beta}(x) \cdot \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}}_{= g'_{\mu\nu}(x')} \cdot dx'^\mu dx'^\nu$$

both metrics describe the same geometry in different coords.

N-transforms, $\frac{N(N+1)}{2}$ entries in $g \rightarrow \frac{N(N-1)}{2}$ independent entries.

careful: not all "ordinary" metric axioms hold

- $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$ symmetric ✓
- $ds^2 = 0$ ~ light-cones

general relativity manifolds + coordinates

o use the metric to do geometry on a manifold

- length of a curve

$$\int_A^B ds = \int_A^B \sqrt{|g_{\mu\nu} dx^\mu dx^\nu|} = \int_A^B d\lambda \sqrt{\left|g_{\mu\nu} \frac{dx^\mu}{d\lambda} \cdot \frac{dx^\nu}{d\lambda}\right|}$$

introduce $|\ldots|$ for pseudo-Riemannian

with a parameter λ that describes a curve from A to B

- areas

start with a diagonal metric:

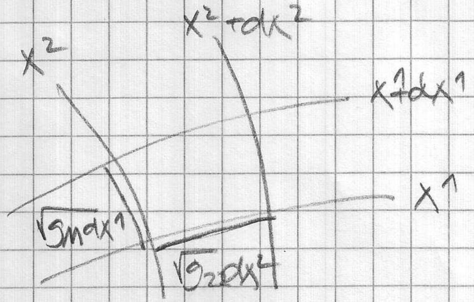
$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \ldots \qquad \text{"orthogonal coordinates}$$

construct area on the (x^1, x^2)-surface:

$\rightarrow$ bounded by lines of constant x^1, x^1+dx^1) (x^2, x^2+dx^2)

$\rightarrow$ length $ds = \sqrt{g_{\mu\mu}}\, dx^\mu$

$$dA = \sqrt{|g_{11} g_{22}|}\, dx^1 dx^2$$



- volumes

in analogy:

$$dV = \sqrt{|g_{11} g_{22} g_{33}|}\, dx^1 dx^2 dx^3$$

and in more than 3 dimensions

$$dS = \sqrt{|g_{11} \cdot \ldots \cdot g_{nn}|}\, dx^1 \cdot \ldots \cdot dx^n$$

if the metric is non-diagonal, one needs to use the determinant (we'll come to that) general relativity : local cartesian coordinates

o local cartesian coordinates

Riemannian manifolds are locally cartesian

(glimpse at the equivalence principle: GR $\to$ SR).

in general, it's not possible to transform coordinates $x^\mu \to x'^\mu$ such that $g_{\mu\nu}(x)$ is brought into Minkowski-form

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \longrightarrow \eta_{\mu\nu}\, dx'^\mu dx'^\nu$$

at every point of the manifold. (again, equivalence!).

exception: high degrees of symmetry (or not!).

why? $\frac{N(N+1)}{2}$ entries in the metric, $g_{\mu\nu} = g_{\nu\mu}$

but only N coordinate transforms.

o locally Minkowskian form of any metric:

find a transformation $x^\mu \to x'^\mu$ that at a single point A the metric $g'_{\mu\nu}$ in the new coordinates satisfies

$$\left.\begin{aligned} g'_{\mu\nu} &= \eta_{\mu\nu} \\ \frac{\partial g'_{\mu\nu}}{\partial x'^\sigma} &= 0 \end{aligned}\right\} \text{ at } A$$

$\to$ local expansion in the vicinity of A.

$$g'_{\mu\nu}(A) = \eta_{\mu\nu} + O\left(|x' - x'_A|^2\right)$$

$\to$ local cartesian coordinates at A.

consider Taylor expansion of the relation $x^\mu(x')$

$$x^\mu(x') = x^\mu(A) + \left.\frac{\partial x^\mu}{\partial x'^\nu}\right|_A \cdot (x'^\nu - x'^\nu_A)$$

$$+ \frac{1}{2} \left.\frac{\partial^2 x^\mu}{\partial x'^\nu \partial x'^\rho}\right|_A (x'^\nu - x'^\nu_A)(x'^\rho - x'^\rho_A)$$

$$+ \frac{1}{3!} \left.\frac{\partial^3}{\partial x'^\nu \partial x'^\rho \partial x'^\sigma}\right|_A (x'^\nu - x'^\nu_A)(x'^\rho - x'^\rho_A)(x'^\sigma - x'^\sigma_A)$$

$+ \ldots$

general relativity : local cartesian coordinates

○ locally minkowskian form of the metric
count # of independent values

- $\left.\frac{\partial x^\mu}{\partial x'^\nu}\right|_A \longrightarrow N^2$ $\qquad g'_{\mu\nu}(A) \longrightarrow \frac{1}{2}N(N+1)$

to make $g'_{\mu\nu}$ Minkowskian, one needs to fix $\frac{1}{2}N(N+1)$ relations, but there are N^2 free values in $\partial x^\mu/\partial x'^\nu$
$\rightarrow$ works always! $\quad g'_{\mu\nu}(A) = \eta_{\mu\nu}$

- $\frac{\partial^2 x^\mu}{\partial x'^\nu \partial x'^\rho} \longrightarrow \frac{1}{2}N(N+1)\cdot N$ $\qquad \frac{\partial g'_{\mu\nu}}{\partial x'^\rho} \longrightarrow \frac{1}{2}N(N+1)\cdot N$

we need to fix $\frac{1}{2}N^2(N+1)$ values in ∂g to be zero, but the transform provides exactly the same number of coefficients in $\partial^2 x^\mu/\partial x'^\nu \partial x'^\rho$
$\longrightarrow$ works always! $\quad \left.\frac{\partial g'_{\mu\nu}}{\partial x'^\rho}\right|_A = 0$

- $\frac{\partial^3 x^\mu}{\partial x'^\nu \partial x'^\rho \partial x'^\sigma} \longrightarrow \frac{1}{6}N^2(N+1)(N+2)$ $\qquad \frac{\partial^2 g'_{\mu\nu}}{\partial x'^\rho \partial x'^\sigma} \longrightarrow \left(\frac{1}{2}N(N+1)\right)^2$

again, we need to fix $\left(\frac{1}{2}N(N+1)\right)^2$-values in $\partial^2 g$ to vanish, but there are only $\frac{1}{6}N^2(N+1)(N+2)$ values provided by the transform in $\partial^3 x^\mu/\partial x'^\nu \partial x'^\rho \partial x'^\sigma$
$\rightarrow$ not enough, not all $\partial^2 g$ vanish

- $\partial^4 x$ and $\partial^3 g$: numbers needed to fix $\partial^{n-1} g = 0$ always exceeds coefficients provided by $\partial^n x$

local cartesian coordinates can be defined with the <u>two</u> conditions $g'_{\mu\nu} = \eta_{\mu\nu}$ and $\frac{\partial}{\partial x'^\rho} g'_{\mu\nu} = 0$

but there will be higher order residual effects

general relativity : tangent spaces

- tangent spaces

  local Cartesian coordinates $\longrightarrow$

  match Minkowskian space locally to the manifold

  local space at A is called tangent space T_A

- arbitrary manifolds + geometry

  - use transformation of measures near linear transforms
  - local tangent-space has a diagonal metric

$$\longrightarrow \quad d^N V = \sqrt{|g_{11} \cdot \ldots \cdot g_{NN}|} \cdot dx^1 \cdot \ldots \cdot dx^N$$

linear transform : $\cdot \; dx^\mu = \underbrace{\frac{\partial x^\mu}{\partial x'^\nu}}_{\equiv J^\mu{}_\nu \text{ Jacobian}} \cdot dx'^\nu$

$$\longrightarrow \quad dx^1 \cdot \ldots \cdot dx^N = \det J \cdot dx'^1 \cdot \ldots \cdot dx'^N$$

general relativity   vector calculus

- fields on manifolds and their transformations
  - scalar field on the manifold : $\phi(x^\mu)$

    $\phi(x^\mu) \to \phi'(x^{\mu'}) = \phi(x^\mu)$ now $x^\mu \to x'^\mu$.

    scalar ~ no internal degree of freedom

  - vector fields

    define vector to be in the tangent-space T_A at A
    - inherits geometry from manifold (like orthogonality through $g_{\mu\nu}(A)$)
    - different points have different tangent-spaces! (a vector $\overrightarrow{AB}$ does not make much sense)

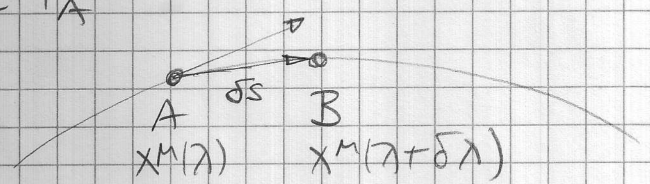
  - tensor fields

    define through cartesian products of Tangent spaces.

- tangent vectors to a curve

  curve C through an N-dimensional manifold : $x^\mu(\lambda)$

  $$\vec{t} = \lim_{\delta\lambda \to 0} \frac{\delta\vec{s}}{\delta\lambda} \in T_A$$



  $\vec{t}$ ~ directional derivative along the curve

  $\delta\vec{s}$ becomes a local quantity and $\in T_A$ in the limit $\delta\lambda \to 0$.

- basis vectors and their construction

  local vector $\vec{v} \in T_A$ at point A

  basis vectors $\vec{e}_\mu$ that span T_A at A

  $$\vec{v}(x) = v^\mu(x) \cdot \vec{e}_\mu(x) = v_\mu(x)\, \vec{e}^\mu(x)$$

  covariant components; basis; contravar. components; dual basis

general relativity

- basis vectors $\vec{e}_\mu$

  dual basis: $\vec{e}^{\mu} \cdot \vec{e}_\nu = \delta^\mu{}_\nu$ at each point A

  $\hookrightarrow$ reciprocal system of vectors

  check out the variation of $\delta\vec{s}$ at A if only x^μ is changed:

  $$\vec{e}_\mu = \lim_{\delta x^\mu \to 0} \frac{\delta \vec{s}}{\delta x^\mu} \sim \text{previous basis for } T_A$$

  $\rightarrow$ $d\vec{s} = \vec{e}_\mu dx^\mu$ joins two points at x^μ and $x^\mu + dx^\mu$

  $$\rightarrow (ds)^2 = d\vec{s} \cdot d\vec{s} = \vec{e}_\mu dx^\mu \cdot \vec{e}_\nu \cdot dx^\nu$$
  $$= \underbrace{\vec{e}_\mu \cdot \vec{e}_\nu}_{\equiv g_{\mu\nu}(x)} dx^\mu dx^\nu$$
  metric

  $\rightarrow$ defines scalar product

  $$\vec{v} \cdot \vec{w} = v^\mu \cdot \vec{e}_\mu \cdot w^\nu \cdot \vec{e}_\nu = g_{\mu\nu} \cdot v^\mu \cdot w^\nu$$

  in analogy for the dual basis:

  $$g^{\mu\nu}(x) = \vec{e}^{\mu} \cdot \vec{e}^{\nu} \quad , \neq g_{\mu\nu} \; !$$

- orthonormal basis vectors $\hat{e}_\mu$

  $\hat{e}_\mu \hat{e}_\nu = \eta_{\mu\nu}$ ~ minkowski

- raising and lowering indices

$$\vec{v} \cdot \vec{w} = \left\{ \begin{array}{l} v^\mu \vec{e}_\mu w^\nu \vec{e}_\nu = g_{\mu\nu} \cdot v^\mu \cdot w^\nu \\ v^\mu \vec{e}_\mu \cdot w_\nu \vec{e}^\nu = g_\mu{}^\nu v^\mu w_\nu \\ v_\mu \vec{e}^\mu \cdot w^\nu \vec{e}_\nu = g^\mu{}_\nu v_\mu w^\nu \\ v_\mu \vec{e}^\mu \cdot w_\nu \vec{e}^\nu = g^{\mu\nu} v_\mu w_\nu \end{array} \right\} = v^\mu w_\mu = v_\mu w^\mu$$

$\rightarrow$ $g_{\mu\nu} v^\mu = v_\nu$ lowering

$g^{\mu\nu} v_\mu = v^\nu$ raising

$$\longrightarrow v^\mu = v^\sigma \delta_\sigma{}^\mu = g^{\mu\nu} v_\nu = \underbrace{g^{\mu\nu} g_{\nu\sigma}}_{= \delta^\mu{}_\sigma} v^\sigma$$

$g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$

general relativity   vector calculus

○ coordinate transforms

x^μ, with basis $\vec{e}_\mu$ $\longrightarrow$ x'^μ with new basis $\vec{e}'_\mu$

look at infinitesimal vector $d\vec{s} = \vec{e}_\mu dx^\mu \equiv \vec{e}'_\mu dx'^\mu$

with transform $dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$, from $x^\mu \to x'^\mu$

$\rightarrow \vec{e}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \cdot \vec{e}_\nu$, or equivalently $\vec{e}'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \vec{e}^\nu$

$\rightarrow$ transform of a vector

$\vec{v} = v^\mu \vec{e}_\mu = v'^\mu \cdot \vec{e}'_\mu$ in the bases $\vec{e}_\mu$ and $\vec{e}'_\mu$

- contravariant $v'^\mu = \vec{e}'^\mu \cdot \vec{v} = \frac{\partial x'^\mu}{\partial x^\nu} \vec{e}^\nu \cdot \vec{v} = \frac{\partial x'^\mu}{\partial x^\nu} \cdot v^\nu$

- covariant $v'_\mu = \vec{e}'_\mu \vec{v} = \frac{\partial x^\nu}{\partial x'^\mu} \cdot \vec{e}_\nu \cdot \vec{v} = \frac{\partial x^\nu}{\partial x'^\mu} \cdot v_\nu$

$\longrightarrow$ transform with the Jacobian and the inverse

○ invariant properties of vectors

$\vec{v} \cdot \vec{w} = v^\mu w_\mu = v_\mu w^\mu = g_{\mu\nu} v^\mu w^\nu = g^{\mu\nu} v_\mu w_\nu$

- length of a vector $|\vec{v}| = v = \sqrt{|g_{\mu\nu} v^\mu v^\nu|}$

$|v| = 0$ possible for $\vec{v} \neq \vec{0}$, null vectors

photon 4-wave vectors are null vectors

they are orthogonal to themselves

- angle $\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}$

but be careful: $\cos\theta > 1$ in a pseudo-Riemannian manifold $\longrightarrow$ $\theta \in \mathbb{C}$.