Übungsaufgaben 8

1.) a) Z: $L(\gamma) = \int_a^b \sqrt{f(\phi)^2 + f'(\phi)^2}\, d\phi$ für $f \in C^1$.

Bew.: Da $f \in C^1$, ist $\gamma \in C^1$.

$$\gamma'(\phi) = (f'(\phi)\cos\phi - f(\phi)\sin\phi,\ f'(\phi)\sin\phi + f(\phi)\cos\phi)$$

$$\Rightarrow \|\gamma'(\phi)\|^2 = (\ldots) \underset{\sin^2\phi + \cos^2\phi = 1}{=} f(\phi)^2 + f'(\phi)^2$$

$$\xRightarrow[\text{Vorlesung}]{\gamma \in C^1} L(\gamma) = \int_a^b \|\gamma'(\phi)\|\, d\phi = \int_a^b \sqrt{f(\phi)^2 + f'(\phi)^2}\, d\phi \qquad \square$$

b) $f(\phi) = a(1+\cos\phi)$, $f'(\phi) = -a\sin\phi$

$$\overset{a)}{\Rightarrow} L(\gamma) = \int_0^{2\pi} \sqrt{a^2(1+\cos\phi)^2 + a^2\sin^2\phi}\, d\phi$$

$$= a\int_0^{2\pi} \sqrt{2(1+\cos\phi)}\, d\phi \qquad \text{mit } \cos^2\phi + \sin^2\phi = 1$$

Add. thm.
$\cos(\frac{\phi}{2} + \frac{\phi}{2}) = \cos^2(\frac{\phi}{2}) - \sin^2(\phi/2)$
$= 2\cos^2(\phi/2) - 1$

$$= 2a \int_0^{2\pi} |\cos(\phi/2)|\, d\phi$$

Substitution $\phi \to 2\phi$

$$= 4a \int_0^{\pi} |\cos\phi|\, d\phi$$

$\cos\phi \geq 0$ für $\phi \in [0, \frac{\pi}{2}]$
$\cos\phi \leq 0$ für $\phi \in [\frac{\pi}{2}, \pi]$

$$= 4a\left(\int_0^{\pi/2} \cos\phi\, d\phi - \int_{\pi/2}^{\pi} \cos\phi\, d\phi\right)$$

$$= 4a(1-(-1)) = 8a$$

2.) $\gamma(\phi) = (e^\phi\cos\phi,\ e^\phi\sin\phi)$

$\gamma'(\phi) = (e^\phi(\cos\phi - \sin\phi),\ e^\phi(\sin\phi + \cos\phi))$

$$\int_\gamma (x\,dy - y\,dx)$$

$$= \int_a^b \left(e^\phi\cos\phi \cdot e^\phi(\sin\phi + \cos\phi) - e^\phi\sin\phi \cdot e^\phi(\sin\phi + \cos\phi)\right) d\phi$$

$$= \ldots \underset{\cos^2\phi + \sin^2\phi = 1}{=} \int_a^b e^{2\phi}\, d\phi = \frac{1}{2}(e^{2b} - e^{2a})$$

3.) $\gamma'(t) = (-r\sin t,\ r\cos t,\ h)$, $\|\gamma'(t)\|^2 = r^2 + h^2$

a)
$$\int_\gamma \left((x^2-y^2)\,dx + 3z\,dy + 4xy\,dz\right)$$
$$= \int_0^{2\pi} \Big((r^2\cos^2 t - r^2 \underbrace{\sin^2 t}_{=1-\cos^2 t})(-r\sin t) + 3ht\,r\cos t + 4r\cos t\, r\sin t \cdot h \Big)\,dt$$
$$= \int_0^{2\pi} \left(-r^2\cos^2 t\sin t + r^2\sin t + 3hr\, t\cos t + 4hr^2 \cos t \sin t \right) dt$$
$$= \left(\tfrac{1}{3} r^2 \cos^3 t - r^2\cos t + 3hr(t\sin t - \cos t) - 2hr^2\cos^2 t \right)\Big|_{t=0}^{2\pi}$$
$$= 0 \qquad \text{(oder mit Symmetrie-Argumenten)}$$

b)
$$\int_\gamma (x^4+y^4+z^4)\,ds$$
$$= \int_0^{2\pi} \left((r\cos t)^4 + (r\sin t)^4 + (ht)^4 \right) \|\gamma'(t)\|\,dt$$
$$= \ldots = \sqrt{r^2+h^2}\left(\frac{3}{2}\pi r^4 + \frac{32}{5}\pi^5 h^4 \right)$$

Add. thm.:
$$\cos 2t = \cos(t+t) = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$$
$$\Rightarrow \quad \cos^2 t = \tfrac{1}{2}(\cos(2t) + 1)$$
$$\Rightarrow \quad \cos^4 t = \tfrac{1}{4}(\cos(2t)+1)^2 = \tfrac{1}{4}(\cos^2(2t) + 2\cos(2t) + 1) = \tfrac{1}{8}(\cos(4t) + 4\cos(2t) + 3)$$

$$\sin^4 t = (1-\cos^2 t)^2 = \cos^4 t - 2\cos^2 t + 1 = \cos^4 t - \cos 2t$$

Also: $\displaystyle\int_0^{2\pi} \sin^4 t\,dt = \int_0^{2\pi} \cos^4 t\,dt = \frac{3}{4}\pi$