Übungsaufgaben 5

1.) a) $\|v\|_2 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$$\partial_v f(1,0) = \langle \nabla f(1,0), v \rangle \quad , \quad \nabla f(x,y) = \begin{pmatrix} y \cos xy \\ x \cos xy \end{pmatrix}$$

$$= 1 \cdot \cos(0) \cdot v_2$$

$$= \frac{\sqrt{3}}{2}$$

b)

$$v' = \frac{v}{\|v\|_2} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\partial_{v'} f(0,0,1) = \langle \nabla f(0,0,1), v' \rangle \quad , \quad \nabla f(x,y,z) = \begin{pmatrix} 2x_y \\ ze^y \\ e^y \end{pmatrix}$$

$$= 2 \cdot 0 \cdot \frac{1}{\sqrt{2}} + e^0 \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

2.) Z: $\|\nabla f(x)\| = \partial_{\frac{\nabla f(x)}{\|\nabla f(x)\|}} f(x) = \max\limits_{\|v\|=1} \partial_v f(x)$ falls $\nabla f(x) \neq 0$.

Bew.: Sei $v \in \mathbb{R}^n$ mit $\|v\|_2 = 1$.

Da f diffbar in x, gilt:

$$\partial_v f(x) = \langle \nabla f(x), v \rangle \underset{\substack{\text{Cauchy} \\ \text{Schwarz}}}{\leq} \|\nabla f(x)\| \cdot \|v\| = \|\nabla f(x)\|$$

Außerdem gilt:

$$\partial_{\frac{\nabla f(x)}{\|\nabla f(x)\|}} f(x) = \left\langle \nabla f(x), \frac{\nabla f(x)}{\|\nabla f(x)\|} \right\rangle = \frac{\|\nabla f(x)\|^2}{\|\nabla f(x)\|} = \|\nabla f(x)\|$$

□.

Man kann auch noch zeigen, dass das Max. eindeutig ist, indem man verwendet, dass bei der Cauchy-Schwarz-Ungl. genau dann Gleichheit gilt, wenn $\nabla f(x) \parallel v$, also

$$\exists \lambda \in \mathbb{R} : \quad v = \lambda \nabla f(x)$$

$$\underset{\|v\|=1}{\Longrightarrow} \quad v = \frac{\nabla f(x)}{\|\nabla f(x)\|}$$

Z: $\nabla f(x) = 0 \Longrightarrow \partial_v f(x) = 0 \quad \forall v \in \mathbb{R}^n, \|v\|_2 = 1$

Bew.: $\partial_v f(x) = \langle \nabla f(x), v \rangle = \langle 0, v \rangle = 0$

□

3.) $\nabla f(x,y) = \begin{pmatrix} \frac{2y}{(x+y)^2} \\ \frac{-2x}{(x+y)^2} \end{pmatrix}$ $\quad H_f(x,y) = \begin{pmatrix} \frac{-4y}{(x+y)^3} & \frac{2x-2y}{(x+y)^3} \\ \frac{2x-2y}{(x+y)^3} & \frac{4x}{(x+y)^3} \end{pmatrix}$

$$f((1,1)+v) = f(1,1) + \langle \nabla f(1,1), v\rangle + \tfrac{1}{2}\langle v, H_f(1,1)v\rangle + o(\|v\|^2)$$
$$= \tfrac{1}{2}v_1 - \tfrac{1}{2}v_2 - \tfrac{1}{2}v_1^2 + \tfrac{1}{2}v_2^2 + o(\|v\|^2)$$

4.) a) $\nabla f(x,y) = \begin{pmatrix} \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y) \\ \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y) \end{pmatrix} = \begin{pmatrix} \sin y \cdot \sin(2x+y) \\ \sin x \cdot \sin(x+2y) \end{pmatrix}$

$$\nabla f(x,y) = 0 \overset{0<x<\pi}{\underset{0<y<\pi}{\Longrightarrow}} \sin(2x+y) = 0 \ \wedge \ \sin(x+2y) = 0$$

$\left.\begin{matrix} 0 < 2x+y < 2\pi \\ 0 < x+2y < 2\pi \end{matrix}\right\}$ $\Longrightarrow \quad 2x+y = \pi \ \wedge \ x+2y = \pi$

$$\Longrightarrow \quad x-y = 0 \ \wedge \ 3(x+y) = 2\pi$$

$$\Longrightarrow \quad x = y = \pi/3$$

$$\nabla f\left(\tfrac{\pi}{3}, \tfrac{\pi}{3}\right) = \left(\sin\tfrac{\pi}{3} \cdot \sin\pi, \ \sin\tfrac{\pi}{3}\sin\pi\right) = 0$$

Also ist $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ die einzige kritische Stelle von f in U.

$$H_f(x,y) = \begin{pmatrix} 2\sin y\cos(2x+y) & \sin(2x+2y) \\ \sin(2x+2y) & 2\sin x\cos(x+2y) \end{pmatrix}$$

$$H_f\left(\tfrac{\pi}{3}, \tfrac{\pi}{3}\right) = \begin{pmatrix} -\sqrt{3} & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -\sqrt{3} \end{pmatrix} \quad \text{neg. definit} \Rightarrow \text{Maximum}$$

$$f\left(\tfrac{\pi}{3}, \tfrac{\pi}{3}\right) = \frac{3\sqrt{3}}{8}$$

b) $\nabla f(x,y) = \begin{pmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{pmatrix}, \quad \nabla f(x,y) = 0 \Longleftrightarrow (x,y) = (1,1) \vee (x,y) = (0,0)$

$$H_f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \quad \text{indefinit} \Rightarrow \text{keine Extremstelle}$$

$$H_f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \quad \text{pos. def.} \Rightarrow \text{Minimum}$$

$$f(1,1) = -1$$