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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 1

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Due: 9:15h, 20 April 2015

1. (10 points) Gravitational train

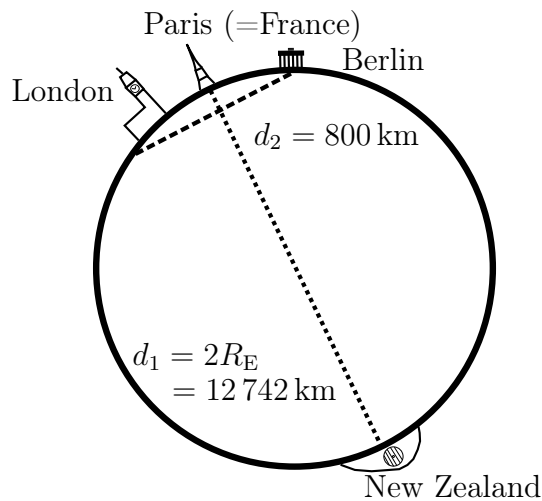


Figure 1: Map of the earth.

In order to save CO_2 emissions from air traffic, the governments of France and New Zealand have decided, very realistically, to bore a tunnel through the centre of the earth, directly connecting their landmasses with a straight tube, through which a train can travel without any friction and solely accelerated by gravitational pull.

- (a) How long does it take for the train to travel one way if Earth's radial density $\rho(r)$ follows
- $\rho(r) = \rho_0 = \text{const.}$, and
 - $\rho(r) = \rho_0/r$,
- for $0 < r \leq R_E$.

- (b) Assume there was a similar tunnel between Berlin and London. How long would that journey take?
- (c) What is the period T of a hypothetical satellite that orbits the Earth on its surface? Why is it exactly the same as the time for the return journey for the train?

2. (10 points) **Lagrangian uniqueness**

Consider the Lagrangian for point mass within a potential V ,

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2}m\dot{q}^2 - V(q), \quad (1)$$

and

- (a) show that the equations of motion (Euler-Lagrange equations) won't change for transformations of the type

$$\mathcal{L} \rightarrow \mathcal{L}'(q, \dot{q}, t) = \alpha\mathcal{L}(q, \dot{q}, t) + \frac{d}{dt}f(q, t), \quad (2)$$

or, in other words, that the action

$$S = \int dt \mathcal{L}(q, \dot{q}, t)$$

is invariant under scaling by an arbitrary number α and gauge transformations which add a total time derivative of an arbitrary function $f(q, t)$ to the Lagrangian.

- (b) What is the physical interpretation of the property you just showed?
- (c) Show that under the infinitesimal transformation $q \rightarrow q + dq$, the Lagrangian gains a contribution of potential energy that can be written in the familiar form Fds . This means that in this case, we don't have homogeneity of space.
- (d) Give an example of a symmetry that the above Lagrangian \mathcal{L} does exhibit, and name the associated conserved quantity.

3. (10 points) **Newton's space rope**

Consider a rope that is connected to the surface of the Earth on the equator. It is affected by the gravitational force

$$F_G(r) = G \frac{Mm}{r^2},$$

as well as a centrifugal force

$$F_C(r) = \frac{mv^2}{r} = m\omega^2 r.$$

- Write the forces as integrals along the rope axis. Assume a constant length-density σ .
- At what point do the two forces cancel out, i.e. how long does the rope have to be to be in an equilibrium?
Use the dimensionless variable $\xi = x/R_E$ to express its length in Earth radii.
- What is the orbital velocity of the outermost point of the rope? (You can assume that the rope is rigid)

4. (10 points) **Keplerian orbits**

Given the Lagrangian

$$\mathcal{L} = \frac{1}{2}m (\dot{r}^2 + r^2\dot{\varphi}^2) - U(r), \quad (3)$$

we're going to find whether Keplerian orbits are closed.

- Name q_i and \dot{q}_i . What assumption about the shape of the orbits has been made? Is this a sensible choice given what you know about planetary orbits?
- What is the physical interpretation of the three summands?
- What is the total Energy of the system $\mathcal{E} = \sum E_i$? Where applicable, write it in terms of the canonical momentum in φ -direction,

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}.$$

- From the above, you can easily arrive at

$$\frac{dr}{dt} = \sqrt{\frac{2}{m} (\mathcal{E} - U(r)) - \frac{p_\varphi^2}{2m^2 r^2}}. \quad (4)$$

Try to express $d\varphi$ in terms of dr ! (Tip: $p_\varphi = mr\frac{d\varphi}{dt}$).

Extra question: What condition needs to apply that φ describes a closed curve?

- (e) From equation 4, we can immediately see that $\dot{r} = 0$ iff

$$\mathcal{E} = U + \frac{p_\varphi^2}{2m^2r^2} = U + \frac{1}{2}mr^2\dot{\varphi}^2.$$

What very special form of orbit do we have in this case of $\dot{r} = 0$? Given

$$U = G\frac{Mm}{r},$$

produce a plot of $\mathcal{E}(r)$ for

$$M = 2 \times 10^{30} \text{ kg},$$

$$m = 1 \times 10^{24} \text{ kg},$$

$$\dot{\varphi} = 2 \times 10^{-7} \text{ s}^{-1}.$$

If done correctly, you should see a minimum at $r \approx 150 \times 10^6$ km. Guess whose potential you just plotted!

5. (5 points) **Extra: Mechanical similarity**

- (a) A team of astronauts lands on Mars. They transmit their first steps on the red planet live on TV and in order to prove that it's not a hoax, one of them uses a simple pendulum of length $l = 1$ m to show to the audience that they are really on Mars.

As the gravitational constant g is supposedly lower on Mars than in a TV studio on Earth, the pendulum should have a larger period $T \propto \sqrt{l/g}$.

Can you, as TV audience, distinguish between a lower gravitational constant g and a slower passage of time?

Could this still be a hoax?

“There is no royal road to geometry.”

– *Euclid*